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THE EFFECT OF REVERBERATION TIME IN THE REPRODUCTION OF SOUNDS

By E. A. Johnson*

In this paper the effect of reverberation on the reproduction of sound is considered.

W. C. Sabine[1] has shown that there is an optimum reverberation time for certain types of music. If the reverberation time is greater than this value the music sounds blurred, and lacks distinctness; if the reverberation time is less than this value the music sounds "dead."

When a sound is reproduced, as in broadcasting of radio programs or as in victrola reproduction, it is desired to give a person listening to this reproduced sound the impression that he is listening to the original sound in a room with the proper acoustics.

In any scheme which involves the reproduction of sound, it is necessary to consider the effect of both the room in which the sound originates, and the room in which it is reproduced on the reverberation time of the reproduced sound.

The purpose of the following paper is to determine from a consideration of the acoustical constants of two such rooms the apparent reverberation time in the room where the sound is being reproduced. By such an analysis it should be possible to indicate what factors ought to be considered in the design of recording rooms and broadcasting studios.

We shall make the same assumptions that are made in obtaining the classical reverberation equation of a single room, namely; that the sound energy is uniformly distributed in both rooms and that the absorption coefficient is independent of amplitude. In

* Contribution from Round Hills Research Laboratory.

[1] W. C. Sabine, Collected Papers.

addition we shall assume that reflection causes no reaction on the reproducing device.

In accordance with the principle of the conservation of energy, the sound energy contained in a room at a given time must equal the amount of sound energy that has been supplied minus the sound energy that has been absorbed, i.e.,

$$\text{Energy} = \text{Income} - \text{Absorption}. \tag{1}$$

Or differentiating:

$$\text{Rate of change of energy} = \text{Rate of supply} - \text{rate of absorption}. \tag{2}$$

In the following discussion we shall use intensity as in the following definition:

Sound intensity of a complex wave is defined as the amount of sound energy which would pass a unit area in unit time for a plane wave with the same sound pressure.[2]

The notation below will be used.

The subscript (1) will refer to quantities having to do with the originating room.

The subscript (2) will refer to quantities having to do with the room in which the sound is reproduced.

The subscript (0) will refer to steady state conditions

And

ρ = sound energy density

E = sound intensity

c = velocity of sound

Σas = open window equivalent of room

V = volume of room

$b = \dfrac{c\Sigma as}{4V}$

t = time from $t = 0$

T = Reverberation time = Time for a sound to reach 10^{-6} of its original intensity.

[2] This definition does not define the intensity in terms of the actual flow of energy in the case of a complex wave. It is adopted because both the ear and the microphone are pressure instruments, and therefore do not distinguish between different rates of energy flow, provided the pressures are equal.

Upon the assumption that the sound energy is uniformly distributed throughout the room, we have the following relation between the sound intensity and energy density, for plane waves:

$$E = \rho c.$$

Whence

$$\rho = \frac{E}{c}. \tag{3}$$

We see that in order to set up a differential equation for the energy we must know the rate of supply and the rate of absorption of energy.

Consider a broadcasting or public address system, in which the original sound energy is converted into electrical energy by means of a microphone, and reconverted into sound energy by a loud speaker. We shall assume that these instruments and the connecting transmission system give a linear response for a single frequency.

The action of a microphone is just the inverse of that of a loud speaker. The first produces a voltage proportional to the excess pressure at its diaphragm, while the second produces an excess pressure at its diaphragm proportional to the voltage impressed upon it. Therefore the action of a microphone and loud speaker, located in two different media and connected by a suitable transmitting system, is to produce at the diaphragm of the loud speaker a pressure proportional to the pressure at the diaphragm of the microphone.[3]

Rayleigh has shown that, for a plane wave, the intensity is proportional to the square of the excess pressure. Therefore it follows that the intensity at the diaphragm of the loud speaker will be proportional to the intensity of a plane wave having the same pressure as that which exists at the diaphragm of the microphone.

Then the rate at which energy will be supplied by the loud

[3] It must be understood that this is only approximately true since we have not considered the effect caused by the relative sizes of the diaphragm compared to the wave length, standing wave patterns in the room, etc.

speaker will be

$$A = S_L E_L.$$

Where

S_L = effective radiating surface of speaker,
E_L = intensity at surface of speaker.

But

$$E_L = \mu_0 E_M.$$

Where

E_M = intensity at microphone,
μ_0 = proportionality factor of system.

Therefore the rate at which energy is being supplied by the loud speaker is given by

$$A = S_L \mu_0 E_M. \tag{4}$$

By the same line of reasoning we can obtain this equation for other reproducing systems such as phonograph recording, and similar sound reproducing methods. We note that in general the proportionality factors will be a function of frequency and therefore the equations we derive will hold only for a single frequency, or a small range of frequencies.

Now consider the case where the microphone is located in one room and the reproducer in another room. We desire to find the reverberation time (T) in the second room, i.e., the time required for a reproduced sound to die to 10^{-6} of its original value, where $t = 0$ is the time referred to the original sound.

We assume that the sound intensity in the first room follows the classical growth and decay equations

$$E_1 = E_{01}[1 - e^{-b_1 t}] \quad \text{Growth} \tag{5}$$

$$E_1 = E_{01} e^{-b_1 t} \qquad \text{Decay.} \tag{6}$$

The rate of supply to the second room in the case of growth is from (4) and (5)

$$A = S_L \mu_0 E_{01}[1 - e^{-b_1 t}].$$

Or if

$$\mu = S_L \mu_0 E_{01},$$

$$A = \mu[1 - e^{-b_1 t}]. \tag{7}$$

And similarly in the case of decay

$$A = \mu e^{-b_1 t}. \tag{8}$$

Buckingham[4] has shown that the rate at which energy is absorbed in a room is given by

$$\frac{\Sigma as}{4} E. \tag{9}$$

where

Σas = open window equivalent.

But, from (1), the energy in the second room at any time is given by

$$\text{energy} = V_2 0_2 = \frac{V_2 E_2}{c}. \tag{10}$$

We are now in a position to write the differential equations of growth and decay in the second room.

From (3), (7), (9) and (10) the growth equation becomes

$$\frac{d}{dt}\left[\frac{V_2 E_2}{c}\right] = \mu\left[1 - e^{-b_1 t}\right] - \frac{\Sigma_2 as}{4} E_2. \tag{11}$$

Or

$$\frac{dE_2}{dt} = \frac{\mu c}{V_2}\left[1 - e^{-b_1 t}\right] - \frac{c\Sigma_2 as}{4V_2} E_2. \tag{12}$$

Letting

$$\frac{\mu c}{V_2} = m$$

and since

$$b_2 = \frac{c\Sigma_2 as}{4V_2},$$

$$\frac{dE_2}{dt} = m\left[1 - e^{-b_1 t}\right] - b_2 E_2. \tag{13}$$

The solution of (13) is

$$E_2 = \frac{m}{b_2} - \frac{me^{-b_1 t}}{b_2 - b_1} + \left(\frac{m}{b_2 - b_1} - \frac{m}{b_2}\right) e^{-b_2 t}, \tag{14}$$

[4] Scientific Papers of the Bureau of Standards No. 506.

which may be written

$$E_2 = \frac{m}{b_2}\left[1 - \frac{e^{-b_1 t}}{1 - \frac{b_1}{b_2}} + \frac{e^{-b_2 t}}{\frac{b_2}{b_1} - 1}\right]. \tag{15}$$

Or

$$E_2 = E_{02}\left[1 - \frac{e^{-b_1 t}}{1 - K} + \frac{K e^{-b_2 t}}{1 - K}\right]. \tag{16}$$

Where

$$E_{02} = \frac{m}{b_2} = \frac{\mu c}{b_2} = \frac{4\mu}{\Sigma a s}.$$

And

$$K = \frac{b_1}{b_2}.$$

When

$$b_1 = b_2, \quad \text{i.e.,} \quad K = 1.$$

The solution of (13) becomes

$$E_2 = E_{02}\left[1 - \frac{t + b}{b} e^{-bt}\right]. \tag{16a}$$

We see from these equations that the growth of the sound intensity will be slower in the second room than in the first.

The equation we are most interested in is the decay equation of intensity in the second room, since the rate of decay, which determines the reverberation time, indicates the confusion to be expected between successive sound impulses.

For decay we obtain from (3), (8), (9) and (10)

$$\frac{d}{dt}\left[\frac{E_2 V_2}{c}\right] = \mu e^{-b_1 t} - \frac{\Sigma_2 a s}{4} E_2. \tag{17}$$

Or

$$\frac{dE_2}{dt} = m e^{-b_1 t} - b_2 E_2. \tag{18}$$

Where m and b_2 are the same as in (12). The solution of this equation is

$$E_2 = \left[E_{02} + \frac{m}{b_1 - b_2}\right] e^{-b_2 t} - \frac{m}{b_1 - b_2} e^{-b_1 t}. \tag{19}$$

If it is assumed that E_{02} is the steady state value as indicated by equation (16), we can write (19) as

$$E_2 = \frac{E_{02}}{K-1}[Ke^{-b_1 t} - e^{-b_2 t}]. \tag{20}$$

When

$$b_1 = b_2, \quad \text{i.e.,} \quad K = 1.$$

The solution of (18) becomes

$$E_2 = E_{02}[1 + bt]e^{-bt}. \tag{20a}$$

In practice, the experimental error in determining the optimum reverberation time is about five percent. We shall assume that an error of one percent will give a sufficiently accurate formula.

Upon this assumption we find that we can neglect the effect of the second room in (20) for the region $.75 > K > 0$.

And (20) becomes

$$E_2 = \frac{E_{02}}{1-K}e^{-b_2 t}. \tag{21}$$

Or solving for the reverberation time

$$T = T_1\left[1 + \frac{1}{13.8} ln \frac{1}{1-K}\right]. \tag{22}$$

Where T is the apparent reverberation time and T_1 is the reverberation time of the first room alone.

We also find that we can neglect the effect of b_1 for the region $\infty > K > 1.32$ and (20) becomes

$$E_2 = \frac{E_{02}K}{K-1}[e^{-b_2 t}]. \tag{23}$$

Solving for T

$$T = T_2\left[1 + \frac{1}{13.8} ln \frac{K}{K-1}\right] \tag{24}$$

We note that we can write K as

$$K = \frac{b_1}{b_2} = \frac{T_2}{T_1}.$$

Finally we have left the region $1.32 > K > .75$ for which no exact solution can be given except when $T_1 = T_2$, and

$$T = 1.21T_1 = 1.21T_2. \tag{25}$$

We can draw some very important conclusions from these equations. Sabine and later authorities have found that music sounds best in a room with a fairly high reverberation time. One second is usually taken as being a good value for music rooms of ordinary size, i.e. of about 5000 cubic feet volume. The reverberation time of an ordinary parlor lies between .1 to .5 seconds. Therefore, if sounds originating in a room with a reverberation time of 1 second are to be reproduced in such a parlor, we find that the reverberation time of the combination will be:

$$T = 1.0\left[1 + \frac{1}{13.8} ln\ 2\right]$$
$$= 1.05 \text{ seconds.}$$

The effect of the second room is so small as to be ignored. Our conclusion is that for reproduction of music, at least, the broadcasting studio or recording room should be designed to give the proper reverberation alone, when the reproducing room is to have a much lower reverberation time.

However, in the case where the reproducing room has a reverberation time greater than three quarters of the reverberation time of the recording room, the reverberation time of the combination will be increased to an appreciable extent. For instance, if a reverberation time of one second is desired and the reproducing room has a reverberation time of .75 seconds, then the recording room should have a reverberation time of about .87 seconds in order to give the apparent reverberation time of 1.0 seconds.

For reproduction in theatres, etc., which have a suitable reverberation time alone, the reverberation time of the recording room should be very small compared to this reverberation time.

Figures (1) and (2) indicate how the design problem may be attacked. From the equations we have developed, if two of the

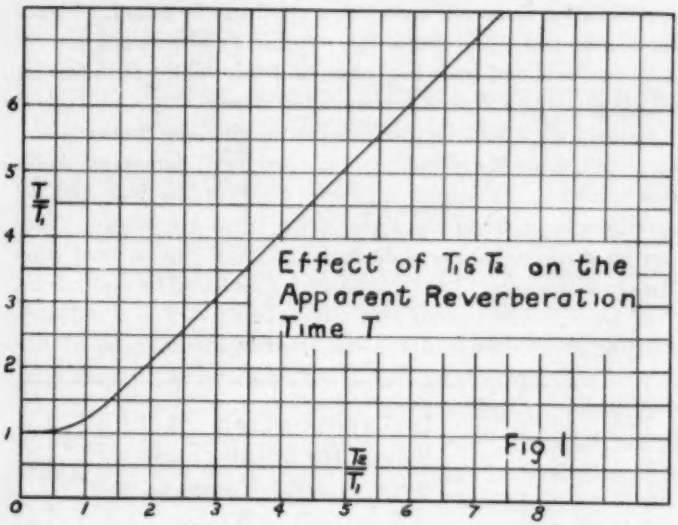

Effect of T_1 & T_2 on the Apparent Reverberation Time T
$\frac{T}{T_1}$
$\frac{T_2}{T_1}$
0 1 2 3 4 5 6 7 8
1 2 3 4 5 6

Fig 1

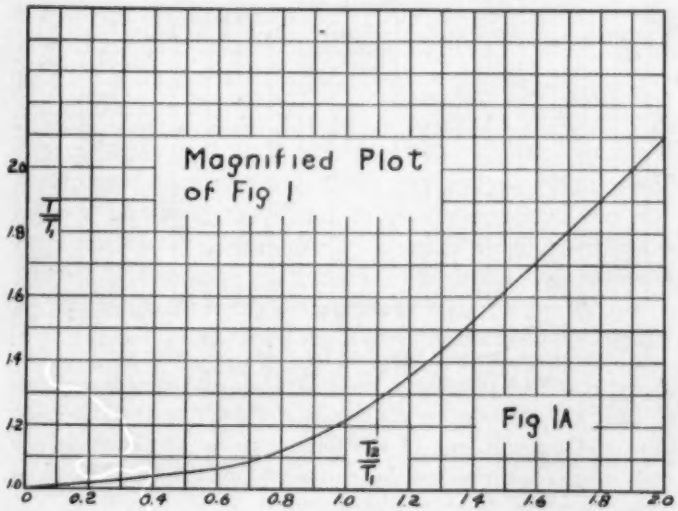
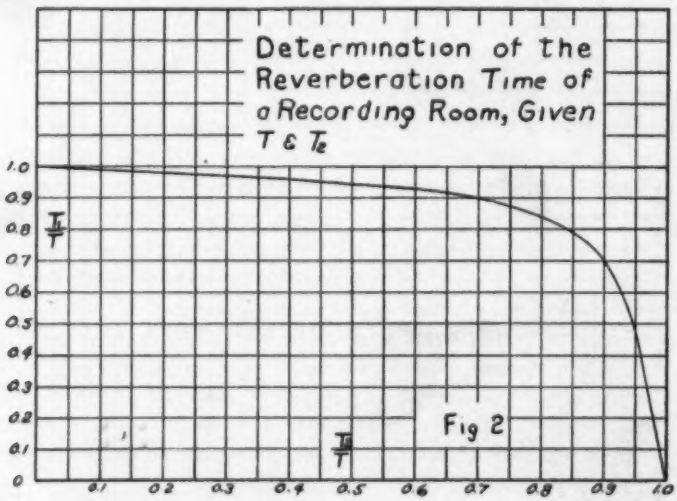

Magnified Plot of Fig 1
$\frac{T}{T_1}$
$\frac{T_2}{T_1}$
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
10 12 14 16 18 20

Fig 1A

reverberation times are known, the third can be found. Figure (1) is a plot of T/T_1 versus T_2/T_1 and if T_1 and T_2 are known, T/T_1 can be found from the curve, and this value multiplied by T_1 will give the apparent reverberation time.

Usually we would be interested in finding the proper reverberation time for the recording room when the optimum reverberation time, i.e., the desired apparent reverberation time, and the reverberation time of the reproducing room are known. This can be done by means of figure (2). T_1/T can be read from the curve, knowing T_2 and T, and then T_1 is found by multiplying T_1/T by T. The intercept of the curve at $T_2/T = 1$ indicates that the reverberation time T can never be less than T_2.



In conclusion, we can say that by using the analysis indicated above, it will be possible to design studios and recording rooms to meet any reproducing conditions. These reproducing conditions will come in one of the three regions mentioned in the derivation of the reverberation equations, and the recording room acoustics can be designed accordingly.

SOME OF THE THERMODYNAMIC PROPERTIES OF AMMONIA*

II. The Joule-Thomson Effect and the Heat Capacity at Constant Pressure

By James A. Beattie

1. Introduction

The calculation of the Joule-Thomson coefficient μ and the heat capacity at constant pressure C_p or at constant volume C_v of a gas can be made if either heat capacity is known as a function of the temperature for some pressure or density and if a suitable equation of state for the gas is available.

The Joule-Thomson coefficients of air and carbon dioxide, and the heat capacities at constant volume of air, carbon dioxide and nitrogen have been evaluated by Keyes [1] from the Keyes equation of state and compared with the experimental data with good success.

Bridgeman [2] has used the Beattie-Bridgeman equation of state [3] to compute μ and the pressure variation of C_p for air. The calculated values agree well with the values obtained by Roebuck [4] from his Joule-Thomson measurements, which extend from 0° to 280° C. and from 1 to 220 atmospheres. The curve which represents the inversion temperature of the Joule-Thomson effect as a function of the pressure is also well reproduced by the values calculated from the equation of state.

The heat capacity and Joule-Thomson effect of ammonia have

* Contribution from the Research Laboratory of Physical Chemistry of the Massachusetts Institute of Technology, No. 228.

[1] Keyes, *Jour. Amer. Chem. Soc.*, *43*, 1452 (1921); *ibid.*, *46*, 1584 (1924); *ibid.*, *49*, 896 (1927).

[2] Bridgeman, *Phys. Rev.*, *34*, 527 (1929).

[3] Beattie and Bridgeman, *Jour. Amer. Chem. Soc.*, *49*, 1665 (1927); *ibid.*, *50*, 3133 (1928); *Proc. Amer. Acad. Arts and Sci.*, *63*, 229 (1928).

[4] Roebuck, *Proc. Amer. Acad. Arts and Sci.*, *60*, 537 (1925).

been studied at the Bureau of Standards by Osborne, Stimson, Sligh and Cragoe [5] and these measurements correlated by means of an equation of state with the compressibility data of Meyers and Jessup [6] which extend from − 35° to 300° C. and from a specific volume of 1300 to 85.5 ccm. per gram.

Recently the compressibility of ammonia has been measured [7] from 75° to 325° C. and from 95 to 20 ccm. per gram, and the constants of the Beattie-Bridgeman equation of state determined from these data. A second set of constants were determined from the Meyers and Jessup measurements. These two groups are given in Table I. It will be noted that the equation of state based on the Beattie and Lawrence data also represents the Meyers and Jessup data fairly well, indicating that the two sets of measurements agree to 0.1 per cent. or better. The great difference in the values of the constants determined from the Meyers and Jessup measurements is probably due to two causes:

TABLE I

CONSTANTS OF THE BEATTIE-BRIDGEMAN EQUATION OF STATE OF AMMONIA

Units: atmospheres, ccm. per gram, degrees K ($T° K = t° C + 273.13$)

	Beattie and Lawrence Data	Meyers and Jessup Data
Mol. Wt.	17.0311	17.0311
R	4.81824	4.81824
A_0	8250.	3000.
a	10.	51.5
B_0	2.005	0.45
b	11.222	131.
c	280×10^6	360×10^6
Average deviation for B. and L. data	0.077%	———
Average deviation for M. and J. data	0.155%	0.043%

[5] Osborne, Stimson, Sligh and Cragoe, Scientific Papers of the Bureau of Standards, *20*, 65 (1924–1926). See also Cragoe, *Refrigerating Engineering*, *12*, 131 (1925).

[6] Meyers and Jessup, *Refrigerating Engineering*, *11*, 345 (1925).

[7] Beattie and Lawrence, *Jour. Amer. Chem. Soc.*, *52*, 6 (1930).

a slight difference in the trend, which would have a great effect on the constants since the data are at low pressures; and the failure of the equation of state to represent the data near saturation within the experimental error, when the constants are determined from measurements at higher temperatures and pressures.

2. The Equation of State

The calculations made in the later sections involve the use of the Beattie-Bridgeman equation of state:

$$\begin{aligned} p &= \frac{RT(1-\epsilon)}{V^2}[V+B] - \frac{A}{V^2}, \\ A &= A_0(1-a/V), \\ B &= B_0(1-b/V), \\ \epsilon &= c/VT^3, \end{aligned} \tag{1}$$

where R is the gas constant and A_0, a, B_0, b and c are constants characteristic of each gas. The volume V relates to unit mass, one gram or one mole, of gas depending upon the units in which the constants are expressed.

The virial form of the equation is

$$\begin{aligned} pV &= RT + \beta/V + \gamma/V^2 + \delta/V^3, \\ \beta &= RTB_0 - A_0 - Rc/T^2, \\ \gamma &= -RTB_0b + A_0a - RB_0c/T^2, \\ \delta &= RB_0bc/T^2, \end{aligned} \tag{2}$$

in which β, γ, and δ are functions only of the temperature and the equation of state parameters.

For the evaluation of quantities which require differentiation with respect to volume, the following arrangement of the terms of the equation is convenient:

$$\begin{aligned} p &= \psi T - \phi - \Gamma/T^2, \\ \psi &= R/V + RB_0/V^2 - RB_0b/V^3, \\ \phi &= A_0/V^2 - A_0a/V^3, \\ \Gamma &= c\psi/V, \\ &= Rc/V^2 + RB_0c/V^3 - RB_0bc/V^4. \end{aligned} \tag{3}$$

Thus ψ, ϕ and Γ are functions only of the volume and the parameters. Differentiation of these functions with respect to volume will be represented by ψ', ϕ' and Γ' respectively.

The independent variables of the equation of state are V and T. Usually thermodynamic quantities are listed for given values of p and T. In order to compare the calculated with the observed values, the thermodynamic quantity under consideration was evaluated at the appropriate temperatures for a series of specific volumes. The corresponding pressures were computed from the values of β, γ and δ which were already available. The thermodynamic quantity was then plotted against the pressure for each isotherm, and the values read off at the desired even pressures. If more accurate values are wanted, a linear equation through the extreme values and a deviation plot may be used.

3. The Heat Capacity at Constant Pressure

The equations developed in this and the following sections are the same as those obtained by Bridgeman and applied to air. The heat capacities at constant pressure and at constant volume are related by the equation:

$$C_p = C_V + T\left(\frac{\partial p}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_p. \tag{4}$$

The relation

$$\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 p}{\partial T^2}\right)_V \tag{5}$$

can be integrated between the limits $V = \infty$ and the variable finite volume V, giving

$$C_V = C_V^* + \int_\infty^V T\left(\frac{\partial^2 p}{\partial T^2}\right)_V dV. \tag{6}$$

Evaluating the partial derivative in equation (4) from the equation of state (1) and passing to the limit $V = \infty$, we obtain

$$C_V^* = C_p^* - R. \tag{7}$$

Substitution of (6) and (7) into (4) gives:

$$C_p = C_p^* - R + \int_\infty^V T\left(\frac{\partial^2 p}{\partial T^2}\right)_V dV + T\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p . \quad (8)$$

By use of the equation of state in the form (3), we obtain from (8)

$$C_p = C_p^* - R - \int_\infty^V 6\Gamma/T^3 dV - \frac{T(\psi + 2\Gamma/T^3)^2}{\psi' T - \phi' - \Gamma'/T^2} \quad (9)$$

$$= C_p^* - R + \frac{Rc}{VT^3}\left(6 + \frac{3B_0}{V} - \frac{2B_0 b}{V^2}\right)$$

$$+ \frac{R[1 + (B_0 + 2c/T^3)/V + (-B_0 b + 2B_0 c/T^3)/V^2 - 2B_0 bc/T^3 V^3]^2}{1 + 2\beta/RTV + 3\gamma/RTV^2 + 4\delta/RTV^3} . \quad (10)$$

The quantities C_V^* and C_p^* are the values C_V and C_p at zero pressure. They are functions of temperature only and can be evaluated from data on either C_V or C_p for finite densities or pressures by use of one or more of the equations (6), (7) and (10).

4. The Joule-Thomson Coefficient

Substituting the equation of state (3) into the general thermodynamic relation

$$\mu C_p = T\left(\frac{\partial V}{\partial T}\right)_p - V, \quad (11)$$

we obtain:

$$\mu C_p = -\frac{\psi T + 2\Gamma/T^2}{\psi' T - \phi' - \Gamma'/T^2} - V \quad (12)$$

$$= \frac{(-B_0 + 2A_0/RT + 4c/T^3) + (2B_0 b - 3A_0 a/RT + 5B_0 c/T)^2/V - 6B_0 bc/T^3 V^2}{1 + 2\beta/RTV + 3\gamma/RTV^2 + 4\delta/RTV^3} , \quad (13)$$

from which μC_p can be evaluated for any desired volume and temperature. By use of C_p calculated from equation (10), the Joule-Thomson coefficient μ can be obtained.

At zero pressure, equation (13) reduces to

$$\mu_0 = \frac{-B_0 + 2A_0/RT + 4c/T^3}{C_p^*} . \quad (14)$$

For the calculation of C_p and μC_p, the quantities B_0, B_0b, A/RT, A_0a/RT, c/T^3, B_0c/T^3 and B_0bc/T^3 are listed for each isotherm at which the computations are to be made. These quantities are multiplied by the proper coefficient and combined so as to give the various coefficients of $1/V$, $1/V^2$ and $1/V^3$ which are listed for each isotherm. These coefficients are then multiplied by the values of $1/V$, $1/V^2$ and $1/V^3$ for each selected specific volume and C_p or μC_p calculated, the corresponding pressure being also computed. The time required for the calculations is not great if the work is done in a systematic manner.

The various terms of equations (10) and (13) which involve the equation of state parameters were first computed using the values of the parameters in compressibility units. These results were then multiplied by the appropriate factors to reduce them to the units in which the thermal data are given. In Table I the constants for ammonia are given in units of atmospheres, ccm. per gram and degrees Kelvin. The units of the experimental data are: C_p in 20° calories (= 4.183 international joules (v)) per degree centigrade per gram and μC_p in international joules (v) per meter of mercury per gram. The corresponding factors derived from constants given in International Critical Tables are 0.024215 for the terms of equation (10) and 0.13328 for equation (13).

5. Comparison with the Experimental Data

Bridgeman has shown that equations (10) and (13) reproduce the properties of air very satisfactorily over the range experimentally investigated by Roebuck. In Table II is given a comparison of the values of μC_p and μ calculated by use of equations (10) and (13) with those experimentally determined at the Bureau of Standards. The agreement is satisfactory at and above 30° C. but is not good at the three lowest temperatures. The calculated values given in Table II were obtained by use of the equation of state constants determined from the compressi-

TABLE II

COMPARISONS OF THE JOULE-THOMSON COEFFICIENTS FOR GASEOUS AMMONIA CALCULATED FROM THE BEATTIE-BRIDGEMAN EQUATION OF STATE WITH THOSE OBTAINED FROM THE EXPERIMENTAL DATA OF THE BUREAU OF STANDARDS

The values of the constants of the equation of state used are given in Table I; they were obtained from the p–V–T data of Meyers and Jessup.

The temperatures of the measurements varied slightly from point to point; they are given in column one. The calculations were made for the temperatures given in column two.

Temp.	Temp	Pressure	μC_p	μC_p	Deviation	Deviation	C_p	C_p	Deviation	Deviation	μ	μ	Deviation	Deviation
Observations	Calculations		Observed	Calculated	Obs.—calcd.		Observed	Calculated	Obs.—calcd.		Observed	Calculated	Obs.—calcd.	
°C	°C	Meters Hg.	Joules per meter Hg per gram			%	Joules per °C per gram			%	Degree C per meter Hg			%
144.95	144.95	0.752	2.90	2.98	−0.08	−2.8	2.317	2.319	−0.002	−0.1	1.25	1.29	−0.04	−3.2
144.94	144.95	2.323	2.91	3.00	−0.09	−3.1	2.347	2.351	−0.004	−0.2	1.24	1.28	−0.04	−3.2
110.04	110.05	1.512	3.90	3.83	+0.07	+1.8	2.278	2.281	−0.003	−0.1	1.71	1.68	+0.03	+1.8
110.06	110.05	10.411	4.33	4.12	+0.21	+4.9	2.550	2.566	−0.016	−0.6	1.70	1.61	+0.09	+5.3
70.05	70.05	1.513	5.38	5.28	+0.10	+1.9	2.232	2.234	−0.002	−0.1	2.41	2.36	+0.05	+2.1
52.49	52.50	1.303	5.96	6.15	−0.19	−3.2	2.208	2.207	+0.001	0.0	2.70	2.79	−0.09	−3.3
52.52	52.50	1.518	6.27	6.17	+0.10	+1.6	2.223	2.219	+0.004	+0.2	2.82	2.78	+0.04	+1.4
49.98	52.50	2.308	6.32	6.26	+0.06	+0.9	2.275	2.265	+0.010	+0.4	2.78	2.76	+0.02	+0.7
30.10	30.15	1.527	7.56	7.65	−0.09	−1.2	2.230	2.213	+0.017	+0.8	3.39	3.46	−0.07	−2.1
30.09	30.15	1.532	7.74	7.65	+0.09	+1.2	2.230	2.213	+0.017	+0.8	3.47	3.46	+0.01	+0.3
30.16	30.15	1.760	7.70	7.69	+0.01	+0.1	2.252	2.231	+0.021	+0.9	3.42	3.45	−0.03	−0.9
30.11	30.15	1.815	7.79	7.70	+0.09	+1.2	2.257	2.235	+0.022	+1.0	3.45	3.45	+0.00	+0.0
30.15	30.15	2.308	7.82	7.80	+0.02	+0.3	2.308	2.274	+0.034	+1.5	3.39	3.43	−0.04	−1.2
30.17	30.15	2.326	8.28	7.80	+0.48	+5.8	2.314	2.275	+0.039	+1.7	3.58	3.43	+0.15	+4.2
30.23	30.15	4.060	9.14	8.19	+0.95	+10.4	2.511	2.424	+0.087	+3.5	3.64	3.38	+0.26	+7.1
30.15	30.15	7.110	10.61	8.98	+1.63	+15.4	2.964	2.741	+0.223	+7.5	3.58	3.28	+0.30	+8.4
−5.02	−5.00	0.392	11.41	10.70	+0.71	+6.2	2.117	2.098	+0.019	+0.9	5.39	5.10	+0.29	+5.4
−5.35	−5.00	2.307	14.14	11.62	+2.52	+17.8	2.553	2.346	+0.207	+8.1	5.54	4.95	+0.59	+10.6
−4.98	−5.00	2.478	14.19	11.72	+2.47	+17.4	2.594	2.371	+0.223	+8.6	5.47	4.94	+0.53	+9.7
−12.92	−12.92	1.475	15.16	12.31	+2.85	+18.8	2.449	2.246	+0.203	+8.3	6.19	5.48	+0.71	+11.5
−18.22	−18.22	0.394	14.58	12.43	+2.15	+14.7	2.141	2.091	+0.050	+2.3	6.81	5.94	+0.87	+12.8

bility measurements of Meyers and Jessup; when the constants derived from the Beattie and Lawrence p–V–T data are used, the deviations are larger.

The Meyers and Jessup constants were used in computing the values given in Table II because their measurements have already been correlated with the Joule-Thomson data by Osborne, Stimson, Sligh and Cragoe,[8] who use an equation of state which contains nine characteristic constants and which they state is in agreement with the experimental results of Meyers and Jessup within their estimated limits of accuracy. Their equation is based partly on the p–V–T and partly on the thermal data. The relation for μC_p obtained from their equation of state represents the experimental values about as well as equation (13) at and above 30° C., the average deviation being somewhat greater than that shown in Table II for this range. At the three lower temperatures their relation reproduces the measured values much better than equation (13).

It seems reasonable to conclude that the Joule-Thomson coefficients for ammonia calculated from the Beattie-Bridgeman equation of state agree with measured values within the consistency of the compressibility data with the Joule-Thomson measurements. This statement may not apply to the region close to the saturation line.

In Tables III and IV, the values of C_p of ammonia calculated from equation (10) are compared with the experimental results. For Table III the equation of state constants derived from the Meyers and Jessup p–V–T data were used and for Table IV those from the Beattie and Lawrence data. Below 100° C. neither equation is satisfactory but above 100° C. the results are fairly satisfactory in both cases. Above the critical temperature—about 133° C.—the equation based on the Beattie and Lawrence data gives a good representation of the experimental measurements.

The results exhibited in Tables II to IV are a very severe test of an equation of state. The data are mostly below the critical

TABLE III

COMPARISON OF THE SPECIFIC HEAT AT CONSTANT PRESSURE OF GASEOUS AMMONIA CALCULATED FROM THE BEATTIE-BRIDGEMAN EQUATION OF STATE WITH THE OBSERVED VALUES OF THE BUREAU OF STANDARDS

The values of the constants of the equation of state used are given in Table I; they were obtained from the p–V–T data of Meyers and Jessup.

The values of the specific heat at zero pressure used are those given by the Bureau of Standards.

Pressure, atm.		0	2	4	6	8	10	12	14	16	18	20
Temp. °C.		Specific heat in 20° calories per degree Centigrade per gran.										
150	Observed	0.5532	0.5595	0.5660	0.5725	0.5792	0.5860	0.5930	0.6001	0.6073	0.6147	0.6223
	Calculated		.5600	.5672	.5745	.5819	.5895	.5972	.6051	.6132	.6214	.6297
	Obs.—calcd.		− .0005	− .0012	− .0020	− .0027	− .0035	− .0042	− .0050	− .0059	− .0067	− .0074
	% deviation		−0.09	−0.21	−0.35	−0.47	−0.60	−0.71	−0.83	−0.97	−1.09	−1.19
100	Observed	0.5306	0.5414	0.5528	0.5648	0.5774	0.5905	0.6043	0.6188	0.6341	0.6502	0.6674
	Calculated		.5421	.5539	.5663	.5792	.5926	.6066	.6212	.6364	.6523	.6689
	Obs.—calcd.		− .0007	− .0011	− .0015	− .0018	− .0021	− .0023	− .0024	− .0023	− .0021	− .0015
	% deviation		−0.13	−0.20	−0.27	−0.31	−0.36	−0.38	−0.39	−0.36	−0.32	−0.22
50	Observed	0.5097	0.5313	0.5562	0.5843	0.6158	0.6510	0.6902	0.7345	0.7854		
	Calculated		.5303	.5521	.5760	.6021	.6301	.6606	.6949	.7328		
	Obs.—calcd.		.0010	.0041	.0083	.0137	.0209	.0296	.0396	.0526		
	% deviation		0.19	0.74	1.42	2.22	3.21	4.29	5.39	6.70		
0	Observed	0.4917	0.5532	0.6392								
	Calculated		.5330	.5810								
	Obs.—calcd.		.0202	.0582								
	% deviation		3.65	9.11								

TABLE IV

COMPARISON OF THE SPECIFIC HEAT AT CONSTANT PRESSURE OF GASEOUS AMMONIA CALCULATED FROM THE BEATTIE-BRIDGEMAN EQUATION OF STATE WITH THE OBSERVED VALUES OF THE BUREAU OF STANDARDS

The values of the constants used are given in Table I; they were obtained from the p–V–T data of Beattie and Lawrence. The values of the specific heat at zero pressure used are those given by the Bureau of Standards.

Pressure, atm.		0	2	4	6	8	10	12	14	16	18	20
Temp. °C.		Specific heat in 20° calories per degree Centigrade per gram.										
150	Observed	0.5532	0.5595	0.5660	0.5725	0.5792	0.5860	0.5930	0.6001	0.6073	0.6147	0.6223
	Calculated		.5593	.5654	.5720	.5786	.5854	.5924	.5995	.6070	.6147	.6225
	Obs.—calcd.		.0002	.0006	.0005	.0006	.0006	.0006	.0006	.0003	.0000	− .0002
	% deviation		0.04	0.11	0.09	0.10	0.10	0.10	0.10	0.05	0.00	−0.03
100	Observed	0.5306	0.5414	0.5528	0.5648	0.5774	0.5905	0.6043	0.6188	0.6341	0.6502	0.6674
	Calculated		.5405	.5512	.5622	.5738	.5860	.5988	.6122	.6265	.6418	.6577
	Obs.—calcd.		.0009	.0016	.0026	.0036	.0045	.0055	.0066	.0076	.0084	.0097
	% deviation		0.17	0.29	0.46	0.62	0.76	0.91	1.07	1.20	1.29	1.45
50	Observed	0.5097	0.5313	0.5562	0.5843	0.6158	0.6510	0.6902	0.7345	0.7854	0.8452	0.9173
	Calculated		.5269	.5453	.5653	.5872	.6113	.6378	.6675	.7011	.7391	.7841
	Obs.—calcd.		.0044	.0109	.0190	.0286	.0397	.0524	.0670	.0843	.1061	.1332
	% deviation		0.83	1.96	3.25	4.65	6.10	7.59	9.12	10.73	12.55	14.52
0	Observed	0.4917	0.5532	0.6392								
	Calculated		.5254	.5645								
	Obs.—calcd.		.0278	.0747								
	% deviation		5.03	11.69								

temperature and near the saturation curve. Moreover the isometrics of ammonia deviate more from linearity than those of any of the eleven other gases for which the equation of state constants have been obtained. And the success of the calculation of C_p and μ from an equation of state depends largely on the exactness with which the curvature of the isometrics is represented both as a function of temperature and of volume.

6. Summary

Values of the Joule-Thomson coefficient and pressure variation of the heat capacity at constant pressure of ammonia have been calculated from the Beattie-Bridgeman equation of state and compared with the experimental measurements. The representation of the Joule-Thomson data above 30° and of the heat capacity above 100° C. is satisfactory. At lower temperatures the deviations become large and may be explained as being due partly to inconsistency between the compressibility and thermal data and partly to the failure of the equation to represent all trends in the compressibility data in the region close to the saturation line.

This comparison is a severe test of an equation of state because of the large deviations from the ideal state exhibited by ammonia, and because the greater part of the thermal data for the present comparison is below the critical temperature of ammonia.

SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS[1]

By P. Franklin and C. L. E. Moore

1. Introduction. In a discussion of surfaces in n-space whose homogeneous coördinates satisfy one or more linear partial differential equations of the second order, Segre[2] proved that when the coördinates satisfy two such equations, the surface is either contained in a linear space of three dimensions, or is the developable surface of a twisted curve, including the special limiting cases of cones and cylinders. Segre's argument is largely geometrical, and his classification proceeds from the solutions, rather than from the differential equations. Similar methods have been applied by Sisam[3] to the study of three-spreads whose coördinates satisfy four or more linear partial differential equations of the second order.

In this paper we shall study systems of linear partial differential equations by analytic methods. Our classification depends on properties of the differential equations, and thus, to some extent, is applicable to a system of equations whose solution is not obtainable explicitly. It is found that the number of parabolic equations implied by a given system, either as algebraic consequences or through the integrability conditions, is a decisive factor in the classification. In order to investigate when all the solutions of a system are linearly dependent on a finite number of functions, we develop a lemma on linear dependence of functions of several variables. This lemma is a generalization of the well-known Wronskian condition for functions of one variable, and is related to a theorem of Kellogg,[4]

[1] Presented to the *American Mathematical Society*, October 26, 1929.
[2] C. Segre, *Torino Atti*, vol. 42, p. 571 (1907).
[3] C. H. Sisam, *American Journal of Mathematics*, vol. 33 (1911), p. 97.
[4] O. D. Kellogg, *Zts. f. Math. u. Phys.*, vol. 53 (1914), p. 159.

though slightly more precise in its result, and somewhat weaker in its restrictions on the functions.

With the analytic machinery which we set up, we are able to give a complete discussion of the result of Segre quoted above, as well as some of the analogous results of Sisam, and some less complete results for systems of linear partial differential equations of higher order than the second.

2. **Linear Dependence.** We begin with a test for linear dependence applicable to a set of functions of two variables. Let $z^1, z^2, \cdots, z^s$ be s functions of two variables u and v, and denote partial differentiation with respect to these variables by the subscripts 1 and 2. We define a generalized Wronskian matrix $W^2_{s,m}$ by the equation

$$W^2_{s,m} = \begin{Vmatrix} z^1 & z^2 & \cdots & z^s \\ z_1^1 & z_1^2 & \cdots & z_1^s \\ z_2^1 & z_2^2 & \cdots & z_2^s \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z_{2\cdots 2}^1 & z_{2\cdots 2}^2 & \cdots & z_{2\cdots 2}^s \end{Vmatrix}. \tag{2.1}$$

Each column of this matrix contains as its terms one of the functions, then the two first partial derivatives of this function, the three second partial derivatives, and so on down to the $(m+1)$ partial derivatives of the mth order. Thus the matrix has $(m+1)(m+2)/2$ rows.

We formulate our test for dependence in terms of this generalized Wronskian in

Theorem I. If, throughout some u, v region R the partial derivatives of the s functions z^i up to and including those of the nth order exist and are continuous, and the two matrices $W^2_{s,n}$ and $W^2_{s,n-1}$ are of the same rank k, there being in particular some determinant of the kth order in $W^2_{s,n-1}$ which never vanishes in R, then the s functions z^i are linearly dependent on precisely k of them.

To prove this, we begin by selecting a determinant of the kth order in $W^2_{s,n-1}$ which does not vanish in R. This can be done

by our hypothesis. Let the rows of this determinant be $r_1, r_2, \cdots, r_k$, and for simplicity, renumber the variables so that its columns are the first k of $W^2_{s,\,n-1}$ or $\hat{W}^2_{s,\,n}$. Now write down the $n(n+1)/2$ equations

$$\begin{aligned} z^i &= \sum_{j=1}^{k} c^j z^j, \\ z^i_1 &= \sum_{j=1}^{k} c^j z_1{}^j, \\ &\cdots\cdots\cdots \\ z^i_{2\cdots 2} &= \sum_{j=1}^{k} c^j z^j_{2\cdots 2}, \end{aligned} \tag{2.2}$$

where i is an integer $> k$, and $\leq s$. The equations of this set corresponding to the rows $r_1, r_2, \cdots, r_k$ have a non-zero determinant, and accordingly may be solved for the c^j as functions of u and v. Moreover, as $W^2_{s,\,n-1}$ is of rank k, the same will be true of the matrix obtained from it by deleting all but the first k and the ith columns. Thus all the rows of this deleted matrix are linearly dependent on any k independent ones, in particular the rows $r_1, r_2, \cdots, r_k$ and consequently the values of c^j found above will satisfy all the equations (2.2).

But $W^2_{s,\,n}$ is also of rank k, and the argument just used shows that the c^j will also satisfy the $(n+1)$ equations

$$\begin{aligned} z^i_{1\cdots 1} &= \sum_{j=1}^{k} c^j z^j_{1\cdots 1}, \\ &\cdots\cdots\cdots \\ z_{2\cdots 2} &= \sum_{j=1}^{k} c^j z^j_{2\cdots 2}, \end{aligned} \tag{2.3}$$

formed analogously to equations (2.2) from the last $(n+1)$ rows of $W^2_{s,\,n}$. Now differentiate each of the equations (2.2) with respect to u. The terms involving the undifferentiated c^j will cancel with the derivatives of z^i on the left, either because of a following equation of the set (2.2), or one of the equations (2.3). Thus the differentiated equations reduce to

$$\begin{aligned} 0 &= \sum_{j=1}^{k} c_1{}^j z^j, \\ &\cdots\cdots\cdots \\ 0 &= \sum_{j=1}^{k} c_1{}^j z^j_{2\cdots 2}. \end{aligned} \tag{2.4}$$

As the equations of this set corresponding to the rows r_1, r_2, $\cdots$, r_k have a non-zero determinant, it follows from them that the $c_1{}^j$ are all zero. By differentiating the equations (2.2) with respect to v, and reasoning similarly, we may show that the $c_2{}^j$ are all zero. Hence the c^j are constant, and z^i is linearly dependent on $z^1, z^2, \cdots, z^k$. As z^i was any function after the first k, all the z^i are linearly dependent on the first k.

The z^i could not be dependent on less than k of them, as then the colums of $W^2_{*,n}$ would also be so dependent, and its rank could not be k.

Corollary. Conversely, if the s functions z^i are linearly dependent on precisely k of them, and we restrict ourselves to regions in which the minors of the generalized Wronskians either vanish everywhere, or never vanish, there will be some consecutive pair $(n \leq k)$ $W^2_{*,n-1}$ and $W^2_{*,n}$ each of rank k.

For, as the columns of all the $W^2_{*,m}$ are linearly dependent on k of them, their ranks can not be greater than k. As the ranks increase with m and $W^2_{*,0}$ is of rank one, when we increase m from 1 to k, we shall either come to two consecutive ones of rank k, as stated in the corollary, or to two consecutive ones with the same rank, less than k. But in this case, with the restriction on regions mentioned, the theorem would require our s functions to be dependent on less than k, contrary to the hypothesis.

To illustrate the force of our restrictions in the theorem and corollary, we recall the classical examples of functions of one variable. Thus, if

$$(2.5)\qquad \begin{array}{llll} z^1 = 0, & u \leq 0, & z^1 = e^{-1/u^2}, & u > 0, \\ z^2 = 2e^{-1/u^2}, & u < 0, & z^2 = 0, & u \geq 0, \end{array}$$

the generalized Wronskians are all of rank one, but in a region including a portion of the line $u = 0$, the functions are not dependent on less than two. Neither our theorem nor our corollary apply here, since no non-vanishing determinant of the first order exists for the region, and the region is of the type ruled out in the corollary. In a region not including the line $u = 0$, the

generalized Wronskians are of rank one, and the functions are linearly dependent on one, since one of them is zero. This is in agreement with the general result.

As an illustration of the corollary for which no pair of consecutive Wronskians before $W_{s,\,k-1}$ and $W_{s,\,k}$ are each of rank k, we may mention the set composed of

$$z^1 = 1, \quad z^2 = u, \quad \cdots, \quad z^k = u^{k-1} \tag{2.6}$$

and $(s - k)$ functions, each a polynomial in u of the $(k - 1)$st degree. For this set the rank of $W_{s,\,m}$ is $(m + 1)$ for $m = 0, 1, 2, \cdots, (k - 1)$.

For s functions of p variables, $u_1, u_2, \cdots, u_p$, we define a generalized Wronskian matrix $W^p_{s,\,m}$, formed analogously to (2.1). Here each column contains as its terms one of the functions, then the p first partial derivatives of this function, the $p(p + 1)/2$ second partial derivatives, and so on down to the $p(p + 1) \cdots (p + m - 1)/m!$ partial derivatives of the mth order. Thus the matrix has $(p + 1)(p + 2) \cdots (p + m)/m!$ rows. The test for linear dependence in this case is given by

Theorem II. If, throughout some p-dimensional region R_p the partial derivatives of the s functions $z^i(u_1, u_2, \cdots, u_p)$ up to and including those of the nth order exist and are continuous, and the two matrices $W^p_{s,\,n}$ and $W^p_{s,\,n-1}$ are of the same rank k, there being in particular some determinant of the kth order in $W^p_{s,\,n-1}$ which never vanishes in R_p, then the s functions z^i are linearly dependent on precisely k of them.

This may be proved by the same reasoning used for theorem I above, and here, too, we have the

Corollary. Conversely, if the s functions $z^i(u_1, u_2, \cdots, u_p)$ are linearly dependent on precisely k of them, and we restrict ourselves to regions in which the minors of the generalized Wronskians either vanish everywhere, or never vanish, there will be some consecutive pair $(n \leq k)$ $W^p_{s,\,n-1}$ and $W^p_{s,\,n}$ each of rank k.

3. Two Second Order Equations with Two Independent Variables. We shall now discuss the possible solutions of two

linear partial differential equations of the second order with two independent variables. We assume our equations are not both consequences of a single second order equation of similar type. Written out explicitly, our equations are

$$\begin{aligned} \sum_{i,j} a^{ij}z_{ij} + \sum_{i} b^{i}z_{i} + cz = 0, \qquad & (a^{ij} = a^{ji}) \\ \sum_{i,j} A^{ij}z_{ij} + \sum_{i} B^{i}z_{i} + Cz = 0. \qquad & (A^{ij} = A^{ji}) \end{aligned} \tag{3.01}$$

Here the coefficients a^{ij}, A^{ij}, b^{i}, B^{i}, c, C are functions of the two independent variables u and v, the subscripts on z denote differentiation with respect to these variables, and the summations for i and j each run from 1 to 2.

Our method consists in the reduction of our equations to a set of canonical forms by a series of transformations of the dependent and independent variables. The given pair of equations may have no solution except the trivial one $z = 0$. If it has any non-zero solution, $z = x(u, v)$, we may effect the transformation $z = Zx$. As the new equations will be satisfied by $Z = 1$, they will each lack the term in z. Accordingly, we may, without loss of generality, assume $c = C = 0$, and shall do so from now on.

We next try to obtain parabolic second order equations which are consequences of the given pair by taking suitable linear combinations:

$$\sum_{i,j} (pa^{ij} + PA^{ij})z_{ij} + \sum_{i} (pb^{i} + PB^{i})z_{i} = 0. \tag{3.02}$$

This equation will be parabolic, provided the discriminant is zero:

$$(pa^{11} + PA^{11})(pa^{22} + PA^{22}) - (pa^{12} + PA^{12})^{2} = 0. \tag{3.03}$$

As this condition is quadratic in the ratio p/P, there will be two (possibly coincident) values of p/P satisfying it. These values will make equation (3.02) a parabolic equation, satisfied by all solutions of the set (3.01).

If the characteristics of these parabolic equations are distinct, we may take them as new coördinate lines $U = \text{const.}$,

$V = \text{const.}$, and thus reduce them to the form

$$z_{11} = b^1 z_1 + b^2 z_2, \qquad z_{22} = B^1 z_1 + B^2 z_2. \tag{3.04}$$

These equations show that all the solutions of the system are linearly dependent on four, or fewer, linearly independent functions. For, if $W^2_{s,\,2}$ be formed for any s solutions, equations (3.04) show that two of its rows will be linearly dependent on the remaining four, so that its rank r can not be greater than four. Furthermore, the equations obtained from (3.04) by differentiation show that all the derivatives of order higher than the second are linearly dependent on those of the first and second orders, so that the ranks of $W^2_{r,\,k}$, $k > 2$, will be equal to r, the rank of $W^2_{r,\,2}$. If, now, all the solutions are not linearly dependent on fewer than four solutions, we may find some four, z^1, z^2, z^3, z^4, which are linearly independent. By the corollary to theorem I, at least in a suitably restricted region, some consecutive pair of the series $W^2_{4,\,k}$ will be of rank 4. Consequently, by the argument just given, the rank of $W^2_{4,\,2}$ and that of $W^2_{4,\,3}$ will each be four, and the same will be true of $W^2_{5,\,2}$ and $W^2_{5,\,3}$ formed for the four particular solutions, and any fifth. Thus, by theorem I, any (fifth) solution of the system is linearly dependent on the four particular ones.

The parabolic equations which we transformed to (3.04) were found from our original equations by algebraic operations. If there were only one parabolic equation in the pencil (3.02), but a second one could be found by setting up the integrability conditions, and then taking linear combinations, our argument would still apply and show linear dependence. Thus the next case to be considered is that in which the parabolic equations obtained from (3.02) have the same characteristics, and no further parabolic equations are consequences of the system. As before, by a change of coördinates, we may reduce the parabolic equation to the form

$$z_{11} = b^1 z_1 + b^2 z_2. \tag{3.05}$$

The z_{11} term may be removed from the second equation by the use of (3.05), and the presence of a z_{22} term would lead to a second parabolic equation. Consequently, unless the system implies a first order equation, the second equation will be reducible to the form

$$z_{12} = B^1 z_1 + B^2 z_2. \tag{3.06}$$

The condition of integrability

$$z_{112} = z_{121} \tag{3.07}$$

leads to the relation

$$b^1 z_{12} + b^2 z_{22} + b_2{}^1 z_1 + b_2{}^2 z_2 = B^1 z_{11} + B^2 z_{21} + B_1{}^1 z_1 + B_1{}^2 z_2. \tag{3.08}$$

This will lead to a second parabolic equation unless $b^2 = 0$. The remaining derivatives of the second order may be eliminated by using (3.05) and (3.06), and unless the system implies a first order equation, the coefficients of the resulting equation must vanish.

We shall simplify the calculation by a change of variable. We have already shown that $b^2 = 0$, so that equation (3.05) is linear with respect to u, and its solution is a linear combination, with coefficients functions of v, of 1 and some function of u and v. Consequently, if we take this function as a new u, the transformed equation will have 1 and u as solutions, and will reduce to

$$z_{11} = 0. \tag{3.09}$$

The terms on the left of (3.08) are now all zero, and the first order equation obtained by elimination is

$$(B_1{}^1 + B^2 B^1) z_1 + (B_1{}^2 + \{B^2\}^2) z_2 = 0. \tag{3.1}$$

Thus, when no first order equation is implied by the system, we must have

$$B_1{}^1 + B^2 B^1 = 0, \qquad B_1{}^2 + (B^2)^2 = 0. \tag{3.11}$$

If $B^2 = 0$, we see that B^1 must be a function of v alone, and we may assume $B^1 = f'(v)$. Our pair of equations is now

$$z_{11} = 0, \qquad z_{12} = f'(v)z_1. \tag{3.12}$$

The second of these shows that

$$z_1 = A(u)e^{f(v)}, \tag{3.13}$$

and the first one shows that $A(u)$ is a constant. The solution of the system (3.12) is accordingly

$$z = Aue^{f(v)} + B(v). \tag{3.14}$$

By taking $ue^{f(v)}$ as a new variable u, this may be simplified to

$$z = Au + B(v). \tag{3.15}$$

When $B^2 \neq 0$, the second equation (3.11) may be written

$$\frac{-B_1^2}{(B^2)^2} = 1, \tag{3.16}$$

whose integral is

$$1/B^2 = u + f(v). \tag{3.17}$$

By using this, the first equation (3.11) may be integrated to give

$$B^1 = \frac{g(v)}{u + f(v)}. \tag{3.18}$$

In view of the last two equations, equation (3.06) becomes

$$\{u + f(v)\}z_{12} = g(v)z_1 + z_2. \tag{3.19}$$

If the functions $f(v)$ and $g(v)$ are such that

$$f'(v) + g(v) = 0, \tag{3.2}$$

we may take as new variables, $U = u + f(v)$, $V = v$, and the system composed of (3.09) and (3.19) is transformed into

$$z_{11} = 0, \qquad uz_{12} = z_2. \tag{3.21}$$

The solution of this pair of equations is found to be

$$z = A + uB(v). \tag{3.22}$$

When (3.2) is not satisfied, we may take as new variables

$$U = u + f(v)$$

and

$$V = -\int\{f'(v) + g(v)\}dv = -f(v) - \int g(v)dv,$$

which transforms the system composed of (3.09) and (3.19) into

$$z_{11} = 0, \qquad uz_{12} = z_2 - z_1. \tag{3.23}$$

The solution of this pair of equations is

$$z = B(v) + uB'(v). \tag{3.24}$$

It may happen that the roots of (3.03) are coincident, and when the equation (3.02) corresponding to them is set up, all the terms containing second derivatives vanish. This will happen whenever the a^{ij} and A^{ij} of (3.01) are proportional, and these equations are not themselves parabolic. If the b^i and B^i were proportional in the same ratio, our equations would be consequences of one of them, and this case has been excluded. If not, our system implies a first order equation. This case, which has been disregarded in the previous discussion, must now be treated.

If a first order equation is implied by the system, its characteristics may be taken as one of the new variables, so that it may be written as

$$z_1 = 0. \tag{3.25}$$

The system may be satisfied by the general solution of this equation,

$$z = B(v). \tag{3.26}$$

We note that here there is one parabolic equation,

$$z_{11} = 0, \tag{3.27}$$

implied by the system.

If the system is not satisfied by all solutions of (3.25) there will be an additional equation, which, in view of (3.25), will reduce to

$$a^{22}z_{22} + b^2z_2 = 0. \tag{3.28}$$

When $a^{22} \neq 0$, we have a second parabolic equation; while when $a^{22} = 0$, and hence $b^2 \neq 0$, we obtain a second parabolic equation by differentiation. Hence these are cases of linear dependence. In fact, when $a^{22} = 0$, or the ratio b^2/a^{22} involves u, the function $B(v)$ of (3.26) must reduce to a constant, so that all the solutions are linearly dependent on one. When the ratio b^2/a^{22} is a function of v alone, equation (3.28) has as its solution a linear combination of l and some function of v, the coefficients being, possibly, functions of u. But by equation (3.25), these coefficients must be constants, and we have linear dependence on two functions.

In addition to the case of linear dependence, we have found four types of solutions, namely, (3.15), (3.22), (3.24), and (3.26). We may transform (3.22) into (3.15) by putting $Z = z/u$, $U = 1/u$, $V = v$. As the only transformations on the independent variable have been multiplication by a fixed function, we may reverse this by multiplying our final solutions by such a function, and may collect our results in

Theorem III. When a pair of linear, partial differential equations of the second order of the form (3.01), *not both consequences of a single such equation, implies two parabolic equations (either as algebraic consequences, or through the integrability conditions), the solutions are all linearly dependent on four or fewer functions, so that the general solution contains at most four arbitrary constants.*

When only one parabolic equation is implied, and at least one must be, the general solution is reducible, by a transformation of the independent variables, to one of the three forms

$$\begin{aligned} z &= f(u, v)[Au + B(v)], \\ z &= f(u, v)[B(v) + uB'(v)], \\ z &= f(u, v)B(v), \end{aligned} \tag{3.29}$$

where $f(u, v)$ is a fixed function, A is an arbitrary constant, and $B(v)$ is an arbitrary function.

Our results are slightly modified for the case in which the equations contain no z terms, and we collect them in

Theorem IV. When a pair of equations (3.01) *with* $c = C = 0$, *not both consequences of a single such equation, implies two parabolic equations, the solutions are all linearly dependent on four or fewer functions, one of them being unity, so that the general solution contains at most four arbitrary constants.*

When only one parabolic equation is implied, and at least one must be, the general solution is reducible, by a transformation of the independent variables, to one of the four forms

$$
\begin{aligned}
z &= Au + B(v), \\
z &= A + uB(v), \\
z &= B(v) + uB'(v), \\
z &= B(v),
\end{aligned}
\tag{3.3}
$$

where A is an arbitrary constant, and $B(v)$ *is an arbitrary function.*

We may interpret our results geometrically by considering a set of n functions z^i, each of which satisfies a particular system of the kind discussed, as the coördinates of a point in n-space. When we use non-homogeneous coördinates, we apply theorem IV, taking a system with $c = C = 0$. The case of two parabolic equations, or linear dependence on one, two, three or four functions, one of which is a constant, gives for the geometric locus a point, straight line, plane, or surface in three-space respectively. The specialized cases of (3.3) lead, in order, to a cylindrical surface, a conical surface, the developable surface formed by the tangents to a twisted curve in n-space, and the points on a curve. This is in agreement with the result of Segre, who only considered surfaces, and so excluded the loci of lower dimensions. For homogeneous coördinates, we use the system (3.01) as it stands, apply theorem III, and obtain similar results, excepting that here we have the additional case of no solution except the trivial one $z = 0$, without geometric significance; and the case of cones and cylinders need not be differentiated.

As appears from the proof, our theorems must be applied in suitably restricted regions. Thus the system might imply two parabolic equations in some regions, and only one in others, and

the solution may shift from any one case to any other. To show that all the sub-cases of linear dependence may actually occur, we list a set of examples. In each case the solution is obtained directly from the system, as indicated.

The system

$$z_{11} = 0, \qquad z_{12} = z, \tag{3.31}$$

has no solution except $z = 0$, since it implies

$$0 = z_{112} = z_{121} = z_1.$$

Hence

$$z_{12} = 0 \quad \text{and} \quad z = 0.$$

The system

$$z_{11} = z_1, \qquad z_{12} = z_1 + z_2, \tag{3.32}$$

has all its solutions linearly dependent on a constant, since it implies

$$z_{12} = z_{112} = z_{121} = z_{11} + z_{21}.$$

Hence

$$z_{11} = 0 \quad \text{and} \quad z_1 = 0.$$

Hence

$$z_{12} = 0 \quad \text{and} \quad z_2 = 0,$$

and finally $z = A$.

The system

$$z_{11} = 0, \qquad z_{12} = z_2, \tag{3.33}$$

has all its solutions linearly dependent on two functions, since it implies

$$0 = z_{112} = z_{121} = z_{21} = z_{12} = z_2.$$

Hence

$$z = A + Bu.$$

The system

$$z_{11} = e^v z_{12}, \qquad z_{12} = e^v z_{22}, \tag{3.34}$$

has all its solutions linearly dependent on three functions, since it implies

$$e^v z_{221} = z_{121} = z_{112} = e^v z_{122} + e^v z_{12}.$$

Hence

$$z_{12} = 0 \quad \text{and} \quad z_{11} = 0 \quad \text{as well as} \quad z_{22} = 0.$$

Hence, finally,

$$z = A + Bu + Cv.$$

The system

$$z_{11} = 0, \qquad z_{22} = 0, \tag{3.35}$$

has all its solutions linearly dependent on four functions, since the solution is linear in u and in v, and hence must be

$$z = A + Bu + Cv + Duv.$$

The system

$$z_{11} = 0, \qquad z_{12} = uz_1, \tag{3.36}$$

has as its solution an arbitrary function of v, since it implies

$$0 = z_{112} = z_{121} = uz_{11} + z_1 = z_1,$$

and hence

$$z = B(v).$$

For completeness we recall the other three special systems found in the general discussion namely,

$$z_{11} = 0, \qquad z_{12} = 0, \tag{3.37}$$

which has as its solution $z = Au + B(v)$;

$$z_{11} = 0, \qquad uz_{12} = z_2, \tag{3.38}$$

which has as its solution $z = A + uB(v)$; and

$$z_{11} = 0, \qquad uz_{12} = z_2 - z_1, \tag{3.39}$$

which has as its solution $z = B(v) + uB'(v)$.

We observe that a pair of equations of the form (3.01) could be constructed whose coefficients possessed continuous partial derivatives up to any required order (or even were infinitely [5] differentiable) and yet, in different u, v regions, reduced to the

[5] Cf., e. g., *Acta Mathematica*, vol. 47 (1926), p. 371, and the references there given. While the discussion there is for functions of one variable, it may be applied to functions of the form $V(v)\ U(u)$, and the coefficients of the equations (3.31) to (3.39) are all of this type.

coefficients of the various sets (3.31) to (3.39) in succession. This shows the necessity of only applying our theorems literally in suitably restricted regions.

These examples also show that it is not the number of parabolic equations in the pencil (3.02), but rather the number implied by the system when the integrability conditions as well as other possible combinations of algebraic reductions and differentiations are taken into account, that fixes the type of solution. Thus, in the first four examples, (3.31) to (3.34), there is in each case only one parabolic equation in the pencil, but two parabolic equations are implied, $z_{11} = 0$ and $z_{22} = 0$, and we have linear dependence. As a further more extreme example we have the system

$$z_{12} = z_1, \qquad z_{12} = z_2. \tag{3.4}$$

Here there are no parabolic equations in the pencil (3.02), but the system implies

$$z_1 = z_2;$$

hence

$$z_{11} = z_{12}, \qquad z_{22} = z_{12},$$

so that the two parabolic equations

$$z_{11} = z_1 \quad \text{and} \quad z_{22} = z_2$$

are consequences of the system, and we must have linear dependence. The solution of (3.4) is readily found to be

$$z = A + Be^{x+y}.$$

Our discussion of the general differential equations shows that when we do not have linear dependence, the solutions of (3.36), (3.37), (3.38) and (3.39) may be taken as canonical forms to which the solution of any system implying only one parabolic equation is reducible by a multiplication of the independent variable by a fixed function and a change of independent variable. Furthermore, except for (3.36) where some modifications of the system are possible, when the solutions are reduced to our

canonical forms, the systems will also essentially be so reduced, as is seen by eliminating the arbitrary constants and functions from our solutions to form second order equations. For the systems (3.31), (3.32), (3.33) and (3.34) the differential equations are not essentially determined by the solutions, for (3.35) they are. The question naturally arises, however, whether the solutions of these cases may be taken as canonical forms for the solutions of systems implying two parabolic equations. When the only solution of the system is 0, we have the solution of (3.31). In other cases, on dividing the independent variable by one solution, we reduce one of the solutions to unity.

When this is the only independent solution, so that the solution is a constant, we have it in the form of the solution of (3.32). When the solution is a linear combination of l and some function of u and v, on taking this function as a new u, we reduce the solution to that given for (3.33).

When the solution is a linear combination of l and two independent functions of u and v, we may take these functions as new variables u and v, and so reduce the solution to that given for (3.34). When, however, there is a functional relation between the two functions, after taking one of them as a new v, the desired solution takes the form

$$z = A + Bv + Cf(v). \tag{3.41}$$

When we calculate the first and second derivatives, and eliminate the arbitrary constants, we obtain the equations

$$z_1 = 0, \qquad z_{12} = 0, \qquad z_{11} = 0. \tag{3.42}$$

Thus any system of two second order equations satisfied by (3.41) will either imply the first of equations (3.42), and be a consequence of this equation, thus having the same solution as the system (3.36); or it will imply the last two of these equations and thus essentially reduce to the system (3.37). In these cases the solution will be $z = B(v)$ and $z = Au + B(v)$ respectively, and so will admit (3.41) as *part* of the solution, but

the discussion shows that (3.41) can not be the complete solution of any system (3.01). As a functional relation between the other two of three functions after one of them has been reduced to unity by division is equivalent to a functional relation between the ratios of the original three functions, we see that a system of the form (3.01) can have as its complete solution a linear combination of three functions z^1, z^2, z^3 if and only if the determinant derived from the Jacobian of their ratios does not vanish, i. e.,

$$\begin{vmatrix} z^1 & z_1{}^1 & z_2{}^1 \\ z^2 & z_1{}^2 & z_2{}^2 \\ z^3 & z_1{}^3 & z_2{}^3 \end{vmatrix} \neq 0. \tag{3.43}$$

If we attempt to find a system whose solution is a linear combination of 1 and three functions, each pair of them being functionally dependent, after taking one of the functions as a new v, the desired solution takes the form

$$z = A + Bv + Cf(v) + Dg(v). \tag{3.44}$$

When the first and second derivatives are calculated, and the arbitrary constants eliminated, we again obtain equations (3.42), and the argument given in connection with them shows that (3.44) may be *part* of the solution of a system of type (3.36) or (3.37), but can not be the complete solution of any system of the form (3.01). If a linear combination of four functions is reducible to the form (3.44) after one of them has been reduced to unity by division, every pair of ratios of the original functions will be functionally dependent, and the determinant formed for any three analogous to (3.43) will vanish. Thus we have just found as one necessary condition for four functions to be the basis for the solution of a system: the rank of the matrix

$$\begin{Vmatrix} z^1 & z_1{}^1 & z_2{}^1 \\ z^2 & z_1{}^2 & z_2{}^2 \\ z^3 & z_1{}^3 & z_2{}^3 \\ z^4 & z_1{}^4 & z_2{}^4 \end{Vmatrix} \tag{3.45}$$

must be three.

But there is a further restriction. For, when some of the ratios are functionally independent, after reduction of one of the functions to unity, we may take two of the other reduced functions as new variables u and v, and the desired solution takes the form

$$z = A + Bu + Cv + Df(u, v). \tag{3.46}$$

To find whether a system is possible, we differentiate, and eliminate the arbitrary constants from the first and second derivatives and so deduce the relations

$$\begin{aligned} z_{11}f_{22} - z_{22}f_{11} &= 0, \\ z_{11}f_{12} - z_{12}f_{11} &= 0, \\ z_{22}f_{12} - z_{12}f_{22} &= 0. \end{aligned} \tag{3.47}$$

These are not independent, and we take as our system some pair which implies all three. If f_{11}, f_{12}, f_{22} are all different from zero, any pair may be used; if one or two of them vanish there is one such pair. All can not vanish, as then $f(u, v)$ would be linearly dependent on 1, u and v. On forming the pencil (3.02) and its discriminant (3.03), we find that there will be two distinct values of the ratio p/P, whenever the Hessian

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} \neq 0. \tag{3.48}$$

Unless the characteristics of the corresponding parabolic equations were functionally dependent, we could transform to the form (3.04), and using the argument given in connection with it, show that all the solutions were linearly dependent on four; not fewer since 1, u, v and $f(u, v)$ satisfy (3.47) and hence the transforms satisfy the new system. We may avoid the difficulty about the characteristics by reasoning directly from the system, i. e., (3.47). For, if $W^2_{5,2}$ be formed for 1, u, v, $f(u, v)$ and any fifth solution, we see from (3.47) that its rank is exactly four, since three of its fourth order determinants are f_{11}, f_{12} and f_{22}. But when (3.48) is satisfied, the equations formed by differentiating (3.47) may be solved for the third derivatives, and hence

the rank of $W^2_{5,2}$ is the same as that of $W^2_{5,2}$ since the last three rows are dependent on the preceding ones. Thus by theorem I the five solutions are linearly dependent, and (3.46) gives the general solution of the system.

Let us study the system (3.47) in the exceptional case when the Hessian (3.48) vanishes. If $f_{12} = 0$, either f_{11} or f_{22} must also vanish (not both), and our system is essentially (3.37). When $f_{12} \neq 0$, we may take as new variables $U = u$, $V = f_1$. By using the condition that the Hessian vanishes, as well as the relations derived from this by differentiation, we find that the transformed system is equivalent to

$$Z_{11} = 0, \qquad F_2 Z_{12} = F_{12} Z_2, \tag{3.49}$$

where Z is z and F is f in the new variables, to which the subscripts now refer. The first equation has as its solution

$$Z = A(V) + UB(V), \tag{3.5}$$

and the second equation gives a linear relation between $A'(V)$ and $B'(V)$, namely,

$$(F_2 - UF_{12})B'(V) = F_{12}A'(V). \tag{3.51}$$

If the ratio $F_{12}/(F_2 - UF_{12})$ is not independent of U, either $A'(V)$ or $B'(V)$ will have to vanish, and we have a system equivalent to (3.38) or (3.37). If this ratio is independent of U, F_2 will reduce to a linear function of U, with coefficients functions of V, and the equations (3.49) are of the form (3.19), with $g(v) = 0$, and so may be transformed to (3.23), i. e., (3.39). That all of these cases may occur is easily proved, we illustrate for the last case. Let

$$\begin{aligned} z = A + B[g(v) + ug'(v)] + C[h(v) + uh'(v)] \\ + D[f(v) + uf'(v)], \end{aligned} \tag{3.52}$$

be the solution of a system. By eliminating the constants, the system is found to be (3.39). Hence it has a solution containing an arbitrary function. Now, if we transform to new variables

$$U = g(v) + ug'(v), \qquad V = h(v) + uh'(v),$$

the solution (3.52) will become of the form (3.46), and as the transformed system will be essentially (3.47), and will have an arbitrary function in the solution, the Hessian of (3.48) will vanish.

If we start with four functions, z^1, z^2, z^3, z^4, we may reduce one to unity by division, and form the Hessian of one of the ratios with respect to the other two, unless the rank of the matrix (3.45) falls. Except for a factor, non-vanishing in this case, the Hessian is

$$\begin{vmatrix} |z_{11}{}^i z_1{}^i z_2{}^i z^i| & |z_{12}{}^i z_1{}^i z_2{}^i z^i| \\ |z_{12}{}^i z_1{}^i z_2{}^i z^i| & |z_{22}{}^i z_1{}^i z_2{}^i z^i| \end{vmatrix}, \tag{3.53}$$

that is a two rowed determinant, each element of which is a four rowed determinant, of which we have written the ith row. When the rank of (3.45) falls, each of these four rowed determinants are zero, and hence the expression (3.53) vanishes. Conversely, when (3.53) vanishes, either all the elements are zero, and the rank of (3.45) falls, or when the transformation above indicated is made, the Hessian of (3.48) vanishes. Thus the vanishing of (3.53) leads to one of the exceptional cases.

We summarize the results of the discussion just given in:

Theorem V. An arbitrary linear combination of any one or two functions, of any three functions for which the expression of (3.43) *does not vanish, or of any four functions for which the expression of* (3.53) *does not vanish, may be the complete solution of a system composed of two second order equations, implying two distinct parabolic equations.*

If a linear combination of three functions for which (3.43), *or of four functions for which* (3.53) *vanishes is set up, and the constants eliminated, a system will be obtained implying one parabolic equation, whose solution involves an arbitrary function, and includes the original expression as a part.*

If a system of two second order equations is given, we may set up an integrability condition for them. This may be a consequence of the original two equations, or may give a new equation.

In the latter case, it may be impossible to derive further equations by this process, or we may obtain a fourth equation. In this case, combination of the equations would lead to a first order equation. As the number and order of these equations will remain unchanged when the independent variables are changed, or the dependent variable is multiplied by a factor, their character may be determined from the canonical forms we have obtained. This proves that the original system may be classified as follows:

Theorem VI. When the two given second order equations imply no essentially new equations, either there are two parabolic equations in the pencil formed from them, or just one. In the first case, the solution is a linear combination of four functions, for which the determinant of (3.53) *is not zero, and the canonical form is* (3.47). *In the second case, the solution involves one arbitrary function, and one arbitrary constant, the canonical form being* (3.37), (3.38), *or* (3.39).

When the equations imply a new second order equation, but no first order equation, the solution is a linear combination of three functions, for which the determinant of (3.43) *is not zero, and the canonical form is* (3.34).

When the equations imply a first order equation, either this implies the whole system, or some equations not consequences of it are present. In the first case, the solution contains one arbitrary function, and the canonical form is (3.36). *In the second case, the solution is a linear combination of zero, one or two functions, according as the equations not consequences of the first order equation imply an equation of the zeroth, first, or second order. The canonical forms are respectively* (3.31), (3.32) *and* (3.33).

This goes somewhat further than Theorems III and IV in enabling us to distinguish the dimensionality of the solution by an examination of the system; separating the case where the solution is a curve from that where it is a developable surface, as well as the different possible dimensionalities of linear spaces.

As an application of theorem VI, we note that several theorems

given by Appell [6] for a system of two equations of special type follow from this theorem. The equations considered by Appell have analytic coefficients, and his proofs are based on the known existence theorems for such systems.

4. Second order equations with k independent variables. Let us now consider a linear partial differential equation of the second order with k independent variables. That is, an equation of the form

$$\sum_{i,j} a^{ij}z_{ij} + \sum_{i} b^{i}z_{i} + cz = 0, \qquad (a^{ij} = a^{ji}). \tag{4.01}$$

The coefficients a^{ij}, b^{i}, c are functions of the k independent variables $u_1, u_2, \cdots, u_k$ and the subscripts on z denote differentiation with respect to these variables, so that the summations for i and j run from 1 to k. If it is possible to transform this equation into one of the form

$$z_{11} = \sum_{i} B^{i}z_{i} + Cz, \tag{4.02}$$

by a change of independent variables, we shall say it is parabolic. Evidently any further change of independent variables not affecting u_1 will leave the form of (4.02) unchanged. We shall refer to the function taken for u_1 to effect the reduction as the characteristic of the original equation. As the ordinary differential equation:

$$\sum_{i,j} A^{ij}du_i du_j = 0, \tag{4.03}$$

whose matrix is the reciprocal of a^{ij}, i. e.

$$\sum_{s} a^{is}A^{sj} = \delta^{ij}, \tag{4.04}$$

is invariably connected with (4.01) in the sense that when the independent variables are transformed, the matrices are again reciprocal, we see that an equation will be parabolic if the form (4.03) is, except for a factor, the square of an exact differential.

[6] P. Appell, Sur les Fonctions hypergéometriques de plusieurs variables, Memorial des Sciences Mathematiques, III, p. 17, 21.

It takes $k(k-1)/2$ algebraic conditions to make it a square, and $(k-1)(k-2)/2$ integrability conditions to make the differential exact, giving $(k-1)^2$ as the total number of conditions.

If a system of equations of the form (4.01) is given, and it implies k parabolic equations, with functionally independent characteristics, we may take these as new variables, and obtain as consequences of the system k equations of the form (4.02), each solved for a different z_{ii} in terms of the function and its first order derivatives. If these equations be differentiated successively, we shall be able to express all the $(k+1)$st derivatives in terms of those of lower orders. Thus, for any s solutions, the ranks of $W^k_{s,\,k+1}$ and $W^k_{s,\,k}$ will be the same, and, in a suitably restricted region, we must have a linear relation. In view of the relations we are assuming, and those derived by differentiation, only those rows can be independent which correspond to partial derivatives not with respect to any one variable twice, so that there are ${}^kC_m$ of the mth order, or 2^k in all. This establishes:

Theorem VII. If a system of second order equations with k independent variables implies k distinct parabolic equations, (either as algebraic consequences or through the integrability conditions) the solutions are all linearly dependent on 2^k or fewer functions, so that the general solution contains at most 2^k arbitrary constants.

Corollary. When the system contains more than $(k-1)^2$ equations, there will, in general, be linear dependence.

For, we may form from them a pencil with $(k-1)^2$ undetermined ratios, which may usually be determined so that the $(k-1)^2$ conditions for a parabolic equation are fulfilled, and as the degree of the algebraic relation is greater than k, in the non-specialized case there will be k distinct parabolic equations.

When $k=2$, we have the case treated in section 3; when $k=3$, we have that treated by Sisam. Here, a system implying three parabolic equations and nothing further would have

solutions linearly dependent on 8 functions; one additional equation (which could be differentiated once) would reduce the number to 6, while two or three additional equations would reduce the number to 5 and 4 respectively, since here no new relations are obtained by differentiation, except for the first equation added. Thus, when there is linear dependence, a system of 4, 5, or 6 equations has its solutions dependent on 6, 5, or 4 functions respectively. This corresponds, in the geometric interpretation, to linear loci of 5, 4, or 3 dimensions, as stated by Sisam. For 6 equations, there always is linear dependence if no first order differential equation is implied, since the equations may be solved for the second derivatives, and so give the parabolic equations directly.

5. Variations of the nth order with k independent variables. A linear differential equation of the nth order in k independent variables will be parabolic if there exists a transformation of the independent variables which reduces the terms of highest order to a single one, $z_{11\ldots1}$. Here, again, we shall speak of the new u_1 as the characteristic, and the new form will not be essentially changed if the remaining variables are transformed. There will be a certain number of algebraic conditions, as well as $(k-1)(k-2)/2$ integrability conditions. Let the total number of conditions be $P(n, k)$.

If a system of equations of the nth order implies k parabolic equations, with functionally independent characteristics, we may take these as new variables, and obtain k equations, each solved for a different $z_{i i\ldots i}$, in terms of the derivatives of lower order. If these equations be differentiated successively, we shall be able to express all the derivatives of the $(kn-k+1)$st order in terms of those of lower order. Thus, for any s solutions, the ranks of $W^k_{s,\,kn-k+1}$ and $W^k_{s,\,kn-k}$ will be the same, and, in a suitably restricted region, we must have a linear relation. As the number of independent rows is equal to the number of derivatives obtained by using each variable 0, 1, ... or $(n-1)$ times, but no more, the total number is n^k. This establishes:

Theorem VIII. If a system of nth order equations with k independent variables implies k distinct parabolic equations, (either as algebraic consequences or through the integrability conditions) the solutions are all linearly dependent on n^k or fewer functions, so that the general solution contains at most n^k arbitrary constants.

Corollary. When the system contains more than $P(n, k)$ equations, there will, in general, be linear dependence.

INTRODUCTION TO HIGHER DETERMINANTS

By Lepine Hall Rice

This paper gives a brief account of some of the elementary properties of those functions of matrices of symbols with multiple suffixes, which correspond to, and include as special cases, the determinant and permanent of an ordinary matrix.

These higher or p-way determinants and permanents are sums of ordinary or 2-way determinants or permanents. Their products are similarly formed. They have a Laplace development, an Albeggiani development, and a Binet-Cauchy relation. Their compounds exhibit familiar properties. For some important types there is a Law of Complementaries and a Law of Extensible Minors.

The reader will find the terminology of ordinary determinants largely retained. An effort has been made to keep the notation simple. It is hoped to make the text interesting to many who may not have present need of the results.

At the end of the paper will be found references to the literature and a few comments, together with an alphabetical index.

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1. The Matrix and Its Determinants

Any n^p quantities or *elements*

$$A_{\alpha_1\alpha_2\ldots\alpha_p}, \qquad (\alpha_i = 1, 2, \cdots, n; \quad i = 1, 2, \cdots, p),$$

taken as a whole, constitute a *matrix* of *class* p and *order* n:

$$A = \| A_{\alpha_1\alpha_2\ldots\alpha_p} \|. \tag{1.1}$$

This *p-way* matrix consists of n *layers* of *aspect* α_1,

$$\| A_{1\alpha_2\ldots\alpha_p} \|, \quad \| A_{2\alpha_2\ldots\alpha_p} \|, \quad \cdots, \quad \| A_{n\alpha_2\ldots\alpha_p} \|, \tag{1.2}$$

which are $(p-1)$-way matrices of order n. Also A consists of n layers of aspect α_2, and so on up to α_p. Thus in particular the rows and columns of a 2-way or ordinary matrix are its layers.

A *transversal* of A is the product of n elements no two of which lie in the same layer of any aspect. Thus the 2-way matrix has $n!$ transversals and the p-way matrix has $(n!)^{p-1}$ transversals.

When the elements of a transversal stand in any sequence, the n *values*

$$\alpha_i{}^1\alpha_i{}^2 \cdots \alpha_i{}^n \tag{1.3}$$

of each of the *indices* $\alpha_1, \alpha_2, \cdots, \alpha_p$ form a *range*—an even (*positive*) or odd (*negative*) permutation of $12 \cdots n$; and the *sign of the range* is plus or minus in accordance.

Upon any rearrangement of the elements of a transversal, the signs of the ranges either all change or all remain unchanged; hence

The product of the signs of any even number of ranges in a transversal is a fixed sign. (1.4)

Determinant.—In A let an even number of indices be selected and called *signant*, the rest being called *nonsignant*. To each transversal let a sign be prefixed, viz., the product of the signs of the signant ranges in that transversal, thus making a *term*. The sum of these terms is a *determinant* of A.

Each possible selection of an even number of indices fixes a *signancy* and a corresponding determinant. A *full-sign* determinant has as many indices signant as the class will allow, hence there is one full-sign determinant of a matrix of even class while there are p full-sign determinants of a p-way matrix of odd class.

Permanent.—The *permanent* of A is the sum of the transversals each with a positive sign, i. e., all the indices are nonsignant.

We give a few examples, using determinants and permanents of order 2, marking the signant and nonsignant indices with the symbols $\pm$ and $+$ respectively. First, the 2-way or ordinary determinant and permanent,

$$|d_{\alpha\beta}^{\pm\pm}| = d_{11}d_{22} - d_{12}d_{21}, \qquad |d_{\alpha\beta}^{++}| = d_{11}d_{22} + d_{12}d_{21}. \quad (1.5)$$

Next, the three 3-way ("cubic") determinants and the 3-way permanent,

$$|D_{\alpha\beta\gamma}^{+\pm\pm}| = D_{111}D_{222} - D_{112}D_{221} - D_{121}D_{212} + D_{122}D_{211}, \quad (1.6)$$

$$|D_{\alpha\beta\gamma}^{\pm+\pm}| = D_{111}D_{222} - D_{112}D_{221} + D_{121}D_{212} - D_{122}D_{211}, \quad (1.7)$$

$$|D_{\alpha\beta\gamma}^{\pm\pm+}| = D_{111}D_{222} + D_{112}D_{221} - D_{121}D_{212} - D_{122}D_{211}, \quad (1.8)$$

$$|D_{\alpha\beta\gamma}^{+++}| = D_{111}D_{222} + D_{112}D_{221} + D_{121}D_{212} + D_{122}D_{211}. \quad (1.9)$$

Lastly, two 4-way determinants. Writing $\frac{1111}{2222}$, etc., for the transversals, we have for the full-sign and one of the six 2-sign determinants,

$$|D_{\alpha\beta\gamma\delta}^{\pm\pm\pm\pm}| = \tfrac{1111}{2222} - \tfrac{1112}{2221} - \tfrac{1121}{2212} + \tfrac{1122}{2211} - \tfrac{1211}{2122} + \tfrac{1212}{2121} + \tfrac{1221}{2112} - \tfrac{1222}{2111}, \quad (1.10)$$

$$|D_{\alpha\beta\gamma\delta}^{+\pm+\pm}| = \tfrac{1111}{2222} - \tfrac{1112}{2221} + \tfrac{1121}{2212} - \tfrac{1122}{2211} - \tfrac{1211}{2122} + \tfrac{1212}{2121} - \tfrac{1221}{2112} + \tfrac{1222}{2111}. \quad (1.11)$$

When only one determinant of a matrix is considered, the indices may usually be written in any convenient order. We shall without remark thus arrange and rearrange indices in the course of discussion.

2. Decomposition into Determinants or Permanents of Lower Class

Two layers of A of different aspects have in common the elements of a $(p-2)$-way matrix of order n. E. g., a layer of aspect α_1 and a layer of aspect α_2,

$$\| A_{i\alpha_3\ldots\alpha_p} \|, \qquad \| A_{\alpha_1 i\alpha_3\ldots\alpha_p} \|, \quad (2.1)$$

have in common or *intersect in* the *$(p-2)$-way couche* of *(bipartite) aspect* $\alpha_1\alpha_2$

$$\| A_{ij\alpha_3\ldots\alpha_p} \|; \quad (2.2)$$

and in general k layers of different aspects $\alpha_{i_1}, \cdots, \alpha_{i_k}$ intersect in a $(p-k)$-*way couche* of (k-*partite*) *aspect* $\alpha_{i_1} \cdots \alpha_{i_k}$. A 1-way couche consisting of the n elements common to $p-1$ layers of different aspects $\alpha_{i_1}, \cdots, \alpha_{i_{p-1}}$ is called a *file* of aspect $\alpha_{i_1} \cdots \alpha_{i_{p-1}}$ and *direction* α_{i_p}. In a 2-way matrix, rows and columns are both layers and files, and the 2-way determinant has unique properties in consequence.

For illustration the 4-way matrix of the determinants (1.10) and (1.11) is here so written as to exhibit the four 2-way couches of aspect $\alpha\beta$:

$$\| D_{\alpha\beta\gamma\delta} \| = \left\| \begin{array}{cc|cc} D_{1111} & D_{1112} & D_{1211} & D_{1212} \\ D_{1121} & D_{1122} & D_{1221} & D_{1222} \\ \hline D_{2111} & D_{2112} & D_{2211} & D_{2212} \\ D_{2121} & D_{2122} & D_{2221} & D_{2222} \end{array} \right\|. \tag{2.3}$$

If of n couches of a given aspect no two lie in the same layer of any aspect, the couches are *perjunctive*. There are $(n!)^{k-1}$ sets of perjunctive couches of any k-partite aspect.

Such a set may be taken as the layers of a $(p-k+1)$-way matrix of order n.

The $(n!)^{p-k}$ *transversals of such a matrix are transversals of* A, *and the transversals of the* $(n!)^{k-1}$ *such matrices are together all the transversals of* A. (2.4)

For example, perjunctive couches of aspect $\alpha\beta$ from the matrix (2.3) will form the two 3-way matrices

$$\left\| \begin{array}{cc|cc} 1111 & 1112 & 2211 & 2212 \\ 1121 & 1122 & 2221 & 2222 \end{array} \right\|, \qquad \left\| \begin{array}{cc|cc} 2111 & 2112 & 1211 & 1212 \\ 2121 & 2122 & 1221 & 1222 \end{array} \right\|, \tag{2.5}$$

and each has four of the eight transversals of (2.3).

We shall now show first that any determinant of A can be *decomposed* into an algebraic sum of determinants or permanents (as the case may be) of these $(p-k+1)$-way matrices, and then that by repeated decomposition a more general formula is obtained.

Denote the nonsignant and signant indices in the given aspect by α's and β's respectively, and the remaining nonsignant and signant indices by γ's and δ's respectively, and write the p-way

matrix.

$$\| A_{\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \beta_t \gamma_1 \cdots \gamma_u \delta_1 \cdots \delta_v} \|, \quad \begin{cases} t = s+1, \\ r+t = k, \\ u+v = p-k, \end{cases} \tag{2.6}$$

and the determinant

$$|A| = |A_{\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \beta_t \gamma_1 \cdots \gamma_u \delta_1 \cdots \delta_v}^{+ \quad \pm \quad + \quad \pm}|. \tag{2.7}$$

Denote by

$$\| A_{[\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \beta_t] \gamma_1 \cdots \gamma_u \delta_1 \cdots \delta_v} \| \tag{2.8}$$

any one of the $(p-k+1)$-way matrices, $\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \beta_t$ being a *k-partite index* that takes n perjunctive *values*.

(*i*) If t is odd, make the γ's in (2.8) nonsignant, the δ's signant, and the k-partite index *signant on* β_t, that is, give the k-partite range the sign of the β_t-range; prefix to each of the $(n!)^{k-1}$ determinants thus obtained the product of the signs of the remaining β-ranges (if any) in the transversals of that determinant; and take the sum of these *components*:

$$|A| = \sum_{\alpha\beta} \epsilon_{\beta_1} \cdots \epsilon_{\beta_s} |A_{[\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \beta_t] \gamma_1 \cdots \gamma_u \delta_1 \cdots \delta_v}^{\pm \quad + \quad \pm}|. \tag{2.9}$$

The individual symbols $\epsilon_{\beta_1}, \cdots, \epsilon_{\beta_s}$ may be interpreted as the signs of the corresponding ranges in all the transversals of a component when the elements of each transversal are *ordered on* β_t, i. e., so ordered that the β_t-range reads $12 \cdots n$. But the prefixed sign $\epsilon_{\beta_1} \cdots \epsilon_{\beta_s}$ *is*, by (1.4), *independent of any special ordering*.

(*ii*) If t is even (0, 2, $\cdots$), make the γ's nonsignant, the δ's signant, and the k-partite index nonsignant; prefix to each determinant (or permanent if there are no δ's) the product of the signs of the β-ranges (if any) in its transversals; and take the sum of these components:

$$|A| = \sum_{\alpha\beta} \epsilon_{\beta_1} \cdots \epsilon_{\beta_s} |A_{[\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \beta_t] \gamma_1 \cdots \gamma_u \delta_1 \cdots \delta_v}^{+ \quad + \quad \pm}|. \tag{2.10}$$

In both cases every transversal thus finally receives the same sign that it has in $|A|$.

In case (*ii*), if there are no β's and no δ's, we have the decomposition of a permanent.

The two cases are respectively illustrated by (*i*) the decomposition of (1.11) into a sum of determinants whose matrices are (2.5), and (*ii*) the decomposition of (1.10) into an algebraic sum of such determinants:

$$|D^{+\pm+\pm}_{\alpha\ \beta\ \gamma\ \delta}| = \sum_{\alpha\beta} |D^{\pm\ +\pm}_{[\alpha\ \beta]\ \gamma\ \delta}|$$
$$= \left(\begin{smallmatrix}1111\\2222\end{smallmatrix} - \begin{smallmatrix}1112\\2221\end{smallmatrix} + \begin{smallmatrix}1121\\2212\end{smallmatrix} - \begin{smallmatrix}1122\\2211\end{smallmatrix}\right) \qquad (2.11)$$
$$+ \left(\begin{smallmatrix}2111\\1222\end{smallmatrix} - \begin{smallmatrix}2112\\1221\end{smallmatrix} + \begin{smallmatrix}2121\\1212\end{smallmatrix} - \begin{smallmatrix}2122\\1211\end{smallmatrix}\right),$$

$$|D^{\pm\pm\pm\pm}_{\beta_1\beta_2\ \delta_1\delta_2}| = \sum_{\beta} \epsilon_{\beta_1\epsilon_{\beta_2}} |D^{+\pm\pm}_{[\beta_1\beta_2]\ \delta_1\delta_2}|$$
$$= \left(\begin{smallmatrix}1111\\2222\end{smallmatrix} - \begin{smallmatrix}1112\\2221\end{smallmatrix} - \begin{smallmatrix}1121\\2212\end{smallmatrix} + \begin{smallmatrix}1122\\2211\end{smallmatrix}\right) \qquad (2.12)$$
$$- \left(\begin{smallmatrix}2111\\1222\end{smallmatrix} - \begin{smallmatrix}2112\\1221\end{smallmatrix} - \begin{smallmatrix}2121\\1212\end{smallmatrix} + \begin{smallmatrix}2122\\1211\end{smallmatrix}\right).$$

The more general formula mentioned above will now be obtained by applying (2.9) and (2.10) to themselves, and again to the new results, and so on. That is, we take some of the γ's and δ's in the components as a second multipartite index, obtaining components of the components, and again in these we take some of the remaining γ's and δ's as a third multipartite index, and so on until the p original single indices have been reduced to q indices, all multipartite or some multipartite and some unipartite, according as all or less than all are formed into groups. In the following notation all are represented as possibly multipartite, but the notation permits any to be unipartite. We shall only set down the final formula, since it can be immediately verified and it includes all the rest.

Assume that each of the first g of the q indices contains an odd number of originally signant indices and each of the remaining $q - g$ an even number. Denote the jth of the q indices by

$$\alpha_{j_1} \cdots \alpha_{j_{r_j}} \beta_{j_1} \cdots \beta_{j_{s_j}} \beta_{j_{t_j}}, \text{ or by } \begin{matrix}\{\alpha\}\\ j1\\ \cdots\\ jr_j\end{matrix} \begin{matrix}\{\beta\}\\ j1\\ \cdots\\ js_j\end{matrix} \begin{matrix}\beta\\ \\ \\ jt_j\end{matrix} \text{ or } \begin{matrix}\{\alpha\}\\ j1\\ \cdots\\ jr_j\end{matrix} \begin{matrix}\{\beta\}\\ j1\\ \cdots\\ jt_j\end{matrix}. \qquad (2.13)$$

Let $h = g + 1$, then the determinant may be written

$$|A| = |A^{+\ \pm\ +\ \pm\ +\ \pm\ +\ \pm}_{\{\alpha\}\ \{\beta\}\ldots\{\alpha\}\ \{\beta\}\ \{\alpha\}\ \{\beta\}\ldots\{\alpha\}\ \{\beta\}}|, \qquad (2.14)$$
$$\begin{matrix}11 & 11 & g1 & g1 & h1 & h1 & q1 & q1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1r_1 & 1t_1 & gr_g & gt_g & hr_h & ht_h & qr_q & qt_q\end{matrix}$$

and *the general formula of decomposition into q-way components is*

$$|A| = \sum_{\alpha\beta}(\Pi_{\{\beta\}}\epsilon_\beta)\,|A \begin{bmatrix} \{\alpha\} & \{\beta\} & \overset{\pm}{\beta} \\ 11 & 11 & 1t_1 \\ \cdots & \cdots & \\ 1r_1 & 1s_1 \end{bmatrix} \cdots \begin{bmatrix} \{\alpha\} & \{\beta\} & \overset{\pm}{\beta} \\ g1 & g1 & gt_g \\ \cdots & \cdots & \\ gr_g & gs_g \end{bmatrix} \begin{bmatrix} \{\alpha\} & \{\beta\} & \overset{+}{} \\ h1 & h1 & \\ \cdots & \cdots & \\ hr_h & hs_h \end{bmatrix} \cdots \begin{bmatrix} \{\alpha\} & \{\beta\} & \overset{+}{} \\ q1 & q1 & \\ \cdots & \cdots & \\ qr_q & qs_q \end{bmatrix}|, \tag{2.15}$$

where $\sum_{\alpha\beta}$ indicates that each possible set of values of the first multipartite index is to be taken with each possible set of values of the second, and so on, giving

$$(n!)^{r_1+t_1-1} \cdots (n!)^{r_q+t_q-1} = (n!)^{\Sigma r+\Sigma t-q}$$

components each having $(n!)^{q-1}$ of the transversals of $|A|$, or together all the $(n!)^{p-1}$ transversals of $|A|$; and $\Pi_{\{\beta\}}\,\epsilon_\beta$ indicates that the sign of each component is the product of the signs of those $\sum t - g$ β-ranges belonging to the β's in braces.

The theorem may be expressed as follows:

If the indices of a p-way determinant of any signancy be separated into q indices, some or all multipartite, in any manner;

and each of these q indices be made nonsignant, or else signant on an arbitrarily chosen originally signant index, according as it contains an even or an odd number of originally signant indices;

and any selection of perjunctive values of each of the q indices be made and a q-way determinant (or permanent, according to the assigned signancy) of the elements given by these values be formed;

and a sign be prefixed to such determinant (or permanent) which is the product of the signs of the ranges of values of all the originally signant indices except those chosen as above;

then the p-way determinant will equal the sum of all the signed q-way determinants (or permanents) which can be formed by varying the selection mentioned.

And if the indices of a p-way permanent be thus separated into q indices, and any selection of perjunctive values of each of these be made and a q-way permanent of the elements given by these values be formed, then the p-way permanent will equal the sum of all the q-way

permanents which can be made by varying the selection mentioned. (2.16)

It is immediately seen that any transversal of $|A|$ appears in one and only one component and there has the sign that it has in $|A|$.

Thus, and usually in numerous ways, any p-way determinant may be decomposed into 2-way determinants or permanents.

The decompositions from class 3 to class 2 are:

$$|A^{+\pm\pm\pm}_{\alpha\beta\delta}| = \sum_{\alpha\beta}|A_{[\alpha\beta]}{}^{\pm\pm}_{\delta}|, \qquad \text{or} \quad [+\overset{\pm}{\pm}]\overset{\pm}{\pm}. \tag{2.17}$$

$$|A^{\pm\pm\pm+}_{\beta_1\beta_2\gamma}| = \sum_{\beta}\epsilon_{\beta_1}\epsilon_{\beta_2}|A_{[\beta_1\beta_2]}{}^{+}_{\gamma}|, \quad \text{or} \quad [\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{+}. \tag{2.18}$$

$$|A^{+++\pm}_{\alpha_1\alpha_2\gamma}| = \sum_{\alpha}|A_{[\alpha_1\alpha_2]}{}^{+}_{\gamma}|, \qquad \text{or} \quad [++]\overset{+}{+}. \tag{2.19}$$

That is, a 3-way determinant may be decomposed by signant files into a sum of 2-way determinants, or by nonsignant files into an algebraic sum of 2-way permanents.

Continuing in the abridged notation used above, in which the original signant and nonsignant indices are represented by large signs, and the signant and nonsignant indices of the components by small signs, in which the original indices whose ranges determine the prefixed signs are marked by superposed ϵ's, and in which the summation signs are suppressed, we list the five decompositions from class 4 to class 3, and the eight decompositions from class 4 to class 2:

$$[\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{\pm}\overset{\pm}{\pm} \quad [\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{+}\overset{+}{+} \quad [+\overset{\pm}{\pm}]\overset{+}{+}\overset{\pm}{\pm} \quad [++]\overset{+}{\pm}\overset{\pm}{\pm}$$
$$[++]\overset{+}{+}\overset{+}{+} \tag{2.20}$$

$$[\overset{\cdot}{\pm}\overset{\cdot}{\pm}\overset{\pm}{\pm}]\overset{\pm}{\pm} \quad [+\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{+} \quad [++\overset{\pm}{\pm}]\overset{\pm}{\pm} \quad [+++]\overset{+}{+} \tag{2.21}$$

$$[\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{[}\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{[}\overset{\cdot}{\pm}\overset{\cdot}{\pm}]\overset{+}{[}++]\overset{+}{[}+\overset{\pm}{\pm}]\overset{\pm}{[}+\overset{\pm}{\pm}]\overset{\pm}{[}++]\overset{+}{[}++]\overset{+}{} \tag{2.22}$$

3. The Cayley Product (File Product) of Two Determinants

This is based on the ordinary product of two 2-way determinants

$$|a^{\pm\pm}_{\alpha_1\alpha_2}| \cdot |b^{\pm\pm}_{\beta_1\beta_2}| = |\textstyle\sum_i a^{\pm}_{\alpha_1 i}\, b_i{}^{\pm}_{\beta_2}|. \tag{3.1}$$

The *locant* of an element of a matrix is the set of index-values giving the position of the element in the matrix. Similarly there is a locant of a file, of a couche, of a transversal, etc.

If the n^{p-1} files of a chosen signant direction of a p-way determinant of order n be multiplied into the n^{q-1} files of a chosen signant direction of a q-way determinant of order n, and the n^{p+q-2} products be taken as the elements of a $(p+q-2)$-way matrix of order n (the locant of each element being formed of the combined locants of the files whose product it is), and if the signancy of the indices in these locants be continued, then the resulting determinant is the product of the given determinants:

$$|A^{(\pm)s_1\ \pm}_{\alpha_1\cdots\alpha_{p-1}\alpha_p}| \cdot |B^{\pm\,(\pm)s_2}_{\beta_1\beta_2\cdots\beta_q}| = |\textstyle\sum_i A^{(\pm)s_1}_{\alpha_1\cdots\alpha_{p-1}i}\, B_i{}^{(\pm)s_2}_{\beta_2\cdots\beta_q}|. \tag{3.2}$$

The symbol $(\pm)_s$ means that the indices to which it applies are of any assigned signancy; it is read, " of signancy s."

In order to prove the theorem, it is convenient to represent the indices of $|A|$ by $\{\alpha\}\{\alpha'\}\alpha''\alpha'''$, where α''' is α_p, α'' is another signant index, $\{\alpha'\}$ is the remaining group of signant indices (in number even, 0, 2, $\cdots$), and $\{\alpha\}$ is the group of nonsignant indices (if any); and similarly for $\beta'''\beta''\{\beta'\}\{\beta\}$. Now completely decompose $|A|$ and $|B|$, in $|A|$ taking all the indices but α''' as a $(p-1)$-partite index signant on α'', and dealing similarly with $|B|$:

$$|A| = \textstyle\sum(\Pi\epsilon_{\alpha'})\, |A_{[\{\alpha\}\{\alpha'\}\alpha''\eta]\alpha'''}|, \quad |B| = \sum(\Pi\epsilon_{\beta'})\, |B^{\pm}_{\beta'''[\beta'']\{\beta'\}\{\beta\}}|. \tag{3.3}$$

Then, by (3.1),

$$|A| \cdot |B| = \textstyle\sum\sum(\Pi\epsilon_{\alpha'})(\Pi\epsilon_{\beta'})\, |\sum_i A_{[\{\alpha\}\{\alpha'\}\alpha''\eta]i} B_{i[\beta'']\{\beta'\}\{\beta\}}|, \tag{3.4}$$

and this is a complete decomposition of

$$|\Sigma_i A^{+}_{[\alpha][\alpha']}{}^{\pm}_{[\alpha'']i} B^{\pm}_{i[\beta'']}{}^{\pm}_{[\beta']}{}^{+}_{[\beta]}|, \tag{3.5}$$

which is the right member of (3.2).

4. The Laplace Development of a Determinant or Permanent

In the matrix $||A_{\alpha_1 \cdots \alpha_p}||$ of order n, the k^p elements lying at the intersection of k layers of each of the p aspects constitute a *minor matrix* of order k. A single element is a minor matrix of order 1.

The *complement* of a minor matrix of order k is the minor matrix of order $n - k$ lying at the intersection of the remaining $n - k$ layers of each aspect.

More generally, any number f of minor matrices of orders $k_1, \cdots, k_f$ are *perjunctive* if $\sum k = n$ and if no two of them contain elements of the same layer of A of any aspect.

If from each minor matrix of a perjunctive set a transversal be taken, the product of these transversals is a transversal of A. (4.1)

Hence,

The product of the permanents of a perjunctive set of minor matrices of A is equal to a sum of terms of the permanent of A. (4.2)

Now let the n layers of A of any aspect be assigned to f groups containing respectively $k_1, \cdots, k_f$ layers. Consider any transversal of A: its k_i elements in the ith group of layers are the elements of a transversal of one and only one minor matrix of order k_i lying in that group of layers, and the f minor matrices thus determined are perjunctive. Hence,

The sum of the products of the permanents of all perjunctive sets of f minor matrices of orders $k_1, \cdots, k_f$ lying in f fixed groups of $k_1, \cdots, k_f$ layers of any aspect of A, is equal to the permanent of A. (4.3)

For determinants, the only new matter is that of signs. The following properties of a determinant will first be established.

If two layers of the same aspect in the matrix of a determinant be transposed, the determinant is unchanged if the aspect is nonsignant while its sign is changed if the aspect is signant. (4.4)

For, the transversals of the second matrix are the transversals of the first, and the transversal which was

$$A_{\alpha_1^1\alpha_2^1\ldots\alpha_p^1}\cdots A_{\alpha_1^i\alpha_2^i\ldots\alpha_p^i}\cdots A_{\alpha_1^j\alpha_2^j\ldots\alpha_p^j}\cdots A_{\alpha_1^n\alpha_2^n\ldots\alpha_p^n}$$

becomes, by the transposition (for example) of the two layers whose locants are $\alpha_1{}^i$ and $\alpha_1{}^j$,

$$A_{\alpha_1^1\alpha_2^1\ldots\alpha_p^1}\cdots A_{\alpha_1^j\alpha_2^i\ldots\alpha_p^i}\cdots A_{\alpha_1^i\alpha_2^j\ldots\alpha_p^j}\cdots A_{\alpha_1^n\alpha_2^n\ldots\alpha_p^n},$$

and bears the same sign if α_1 is nonsignant but the opposite sign if α_1 is signant.

The *main diagonal* of a matrix consists of the elements of that transversal whose ranges are simultaneously in the natural order.

If the layers of a determinant be so rearranged by successive transpositions that the elements of a given term are brought into the main diagonal, the number of signant transpositions is even or odd according as the term is positive or negative. (4.5)

For, the main diagonal term is a positive term.

A *minor* of a determinant is the cosignant determinant (*cosignant*, of the same signancy) of a minor matrix of the matrix of the determinant. An element is a minor of order 1. A *perjunct* is the product of a perjunctive set of minors. A perjunct is *signed* when there is prefixed to it the sign of that term of the determinant whose elements are the elements of the main diagonals of the minors in the perjunct.

A signed perjunct is equal to a certain number of terms of the determinant. (4.6)

For, take any term whose elements lie in the perjunct, and bring it into the main diagonal in the following manner. First bring the part of it that lies in each minor of the perjunct into the main diagonal of that minor: this will require an even or an odd number of signant transpositions according as the product

of the signs of the parts as terms of the minors is a positive or a negative sign. Then bring the parts into the main diagonal of the determinant: this will require an even or an odd number of signant transpositions according as the sign of the perjunct is positive or negative. Thus the sign of the term in the signed perjunct agrees with its sign in the determinant.

A minor lies *on* a group of layers if it lies in the layers and is of order the same as the number of layers in the group.

From the above theorems,

The sum of all those signed perjuncts of a determinant that lie on a complete set of groups of layers of any given aspect, is equal to the determinant. (4.7)

This sum is a *Laplace development* of the determinant.

If the perjuncts each consist of an element and its complementary minor, we may attach the signs to the minors and shall have a development in the elements of a layer and their *cofactors*. If the perjuncts each consist of two complementary minors, we may attach the signs to the second minors and shall have a development in the minors lying on a group of layers and their *algebraic complements*.

5. The Albeggiani Development

The *sum* of h matrices

$$\| A^1_{\alpha_1 \cdots \alpha_p} \|, \quad \| A^2_{\alpha_1 \cdots \alpha_p} \|, \quad \cdots, \quad \| A^h_{\alpha_1 \cdots \alpha_p} \|, \tag{5.1}$$

of order n, is the matrix

$$A = \| \textstyle\sum_r A^r_{\alpha_1 \cdots \alpha_p} \|, \tag{5.2}$$

each element being the sum of the corresponding elements of the matrices added.

Let $|A|$ be any determinant of (5.2), and let $|A^1|, \cdots, |A^h|$ be the cosignant determinants of the matrices (5.1).

It is possible to develop $|A|$ as an algebraic sum of products of minors of $|A^1|, \cdots, |A^h|$, following Albeggiani's development of a 2-way determinant having polynomial elements.

Take any minor of any A^i-determinant, and with it take from any other A^i-determinant any minor lying on layers, of each aspect, all different in position from the layers of the same aspect on which lies the first minor. Continue to take *conjunctive* minors from different A^i-determinants until the sum of their orders is n. Their product is called a *mixed perjunct* of the A^i-determinants. Their elements in $|A|$ determine a perjunct of $|A|$, whose sign prefixed to the mixed perjunct makes the latter a *signed* mixed perjunct. We include also as a signed mixed perjunct an entire A^i-determinant with a positive sign.

The sum of the signed mixed perjuncts of a set of cosignant determinants is equal to the cosignant determinant whose matrix is the sum of their matrices. (5.3)

Proof. Any term of $|A|$ is the sum of h^n monomials with the same sign. Consider any such signed monomial: those of its elements which are elements of the same A^i-determinant are *all* found in one and only one minor of that determinant, and the minors thus involved are perjunctive; hence the signed monomial is a term of one and only one signed mixed perjunct. Moreover, each signed mixed perjunct consists wholly of signed monomials of $|A|$. This completes the proof.

Various results may be obtained from (5.3) by expressing the elements of any determinant $|A|$ as h-nomials by adding $h - 1$ zeros to each element, and then taking the partial elements in any way as the elements of h matrices whose sum is the matrix of $|A|$.

If in taking the partial elements the original elements of $|A|$ be so distributed that each of the h matrices consists of a minor matrix of $|A|$ and zeros, then *any mixed perjunct vanishes unless its minors lie in such minor matrices*, since otherwise some one of its minors has a layer of zeros. Hence,

If the matrix of a determinant be partitioned into minor matrices in any manner, the determinant is equal to the sum of those signed perjuncts whose minors lie in different minor matrices. (5.4)

The Laplace development appears as a special case.

Theorems for permanents are got by slight changes in the wording of (5.3) and (5.4).

6. The Binet-Cauchy Relation

A matrix may be of different orders in different directions. It is *elongated* if the orders are all the same except one which is greater.

Take a matrix A elongated in the last direction and a matrix B elongated in the first direction:

$$||A_{\alpha_1 \cdots \alpha_p}|| \ (\alpha_1, \cdots, \alpha_{p-1} = 1, \cdots, m;\ \alpha_p = 1, \cdots, n > m), \quad (6.1)$$

$$||B_{\beta_1 \cdots \beta_q}|| \ (\beta_1 = 1, \cdots, n;\ \beta_2, \cdots, \beta_q = 1, \cdots, m), \quad (6.2)$$

and multiply the files of the *last* direction of A into the files of the *first* direction of B to form the *product matrix* C of class $p + q - 2$ and order m:

$$C = ||\textstyle\sum_i A_{\alpha_1 \cdots \alpha_{p-1} i} B_{i \beta_2 \cdots \beta_q}||. \quad (6.3)$$

Let an even number of α's including α_p be signant in (6.1), and an even number of β's including β_1 be signant in (6.2); and give the same signancy to the α's and β's in (6.3).

Any m layers of aspect α_p of A and the same m layers of aspect β_1 of B form the matrices of *corresponding* determinants $|A_\Gamma|$ and $|B_\Gamma|$, of the assigned signancy and of order n, where Γ may have any one of $\binom{n}{m}$ values.

The determinant of C is equal to the sum of the products of corresponding determinants of A and B:

$$|C| = \textstyle\sum_\Gamma |A_\Gamma| \cdot |B_\Gamma|. \quad (6.4)$$

Proof. Make an Albeggiani development of $|C|$, regarding the matrix C as the sum of n matrices C^i, one for each value of i in (6.3). Now every minor of each $|C^i|$ of order greater than 1 vanishes. This is true if true of the minors of order 2, for any minor is susceptible of a Laplace development in minors of order

2 and their complements. Consider then the typical term of a minor of order 2:

$$\pm A_{\alpha_1^l \ldots \alpha_{p-1}^l} {}_i B_{i\beta_2^l \ldots \beta_q^l} \cdot A_{\alpha_1^k \ldots \alpha_{p-1}^k} {}_i B_{i\beta_2^k \ldots \beta_q^k},$$

the minor lying on layers α_1^l and α_1^k of aspect α_1 and so for each aspect. In this minor is also found the term

$$\mp A_{\alpha_1^l \ldots \alpha_{p-1}^l} {}_i B_{i\beta_2^k \ldots \beta_q^k} \cdot A_{\alpha_1^k \ldots \alpha_{p-1}^k} {}_i B_{i\beta_2^l \ldots \beta_q^l},$$

the sign being opposite because among the reversed β-ranges an odd number are signant. The terms of the minor therefore cancel in pairs.

Thus the Albeggiani development of $|C|$ reduces to signed products of m elements one from each of m C^is. Consider those from the first m C^is. They form precisely the Albeggiani development of $|C|$ with $n = m$; but this is the product of $|A_\Gamma|$ and $|B_\Gamma|$ with Γ denoting the selection $12 \cdots m$. Similarly for each selection of m C^is, and the theorem is established.

7. Compounds of Product Matrices and Determinants

A *compound matrix* has matrices for elements. A *compound determinant* and *compound permanent* have either determinants or permanents for elements.

The *mth compound* of a matrix of order n has the minor matrices of order m for elements, arranged, in each direction, in the *normal order* of the m-from-n selections, as

$$123\ 124\ 125\ 134\ 135\ 145\ 234\ 235\ 245\ 345, \qquad (7.1)$$

which may be described as a quasi-alphabetical order.

Two compound matrices are *equal* if the corresponding *subelements* or elements of their elements are equal.

The usual rule for forming the product of a scalar and a matrix, fully applied, multiplies the scalar into every subelement of a compound matrix.

The product of the mth compounds of two matrices of any classes and of order n is equal to the scalar $\binom{n-1}{m-1}$ *multiplied into the mth*

compound of their product:

$$(A)_m(B)_m = \binom{n-1}{m-1}(AB)_m. \tag{7.2}$$

Proof. In the product

$$AB = \| \textstyle\sum_i A_{\alpha_1 \ldots \alpha_{p-1} i} B_{i \beta_2 \ldots \beta_q} \|, \tag{7.3}$$

we denote a minor of order m by

$$\| \sum_{i=1}^{n} A_{\theta_1 \ldots \theta_{p-1} i} B_{i \varphi_2 \ldots \varphi_q} \|, \tag{7.4}$$

each θ and each φ taking m values. If each collection of m values be denoted by the corresponding capital letter, the mth compound of AB is

$$(AB)_m = \left\| \| \sum_{i=1}^{n} A_{\theta_1 \ldots \theta_{p-1} i} B_{i \varphi_2 \ldots \varphi_q} \|_{\Theta_1 \ldots \Theta_{p-1} \Phi_2 \ldots \Phi_q} \right\|', \tag{7.5}$$

each Θ and each Φ taking $\binom{n}{m}$ values.

Again, the mth compounds of A and B are

$$(A)_m = \left\| \| A_{\theta_1 \ldots \theta_{p-1} \theta_p} \|_{\Theta_1 \ldots \Theta_{p-1} \Theta_p} \right\|, \tag{7.6}$$

$$(B)_m = \left\| \| B_{\varphi_1 \varphi_2 \ldots \varphi_q} \|_{\Phi_1 \Phi_2 \ldots \Phi_q} \right\|, \tag{7.7}$$

and their product is

$$(A)_m(B)_m = \left\| \sum_{I=1}^{\binom{n}{m}} \| A_{\theta_1 \ldots \theta_{p-1} \theta_p} \|_{\Theta_1 \ldots \Theta_{p-1} I} \| B_{\varphi_1 \varphi_2 \ldots \varphi_q} \|_{I \Phi_2 \ldots \Phi_q} \right\|, \tag{7.8}$$

each element being a file of $(A)_m$ into a file of $(B)_m$, that is, being a sum of $\binom{n}{m}$ products each a minor of A into a minor of B. First form the products, any one of which is

$$\| \textstyle\sum_j A_{\theta_1 \ldots \theta_{p-1} j} B_{j \varphi_2 \ldots \varphi_q} \|, \tag{7.9}$$

where j takes m values, and then form their sum, in each of whose (sub)elements j takes each value from 1 to n $\binom{n-1}{m-1}$ times, so that

finally

$$(A)_m(B)_m = \left\| \| \binom{n-1}{m-1} \sum_{j=1}^{n} A_{\theta_1 \dots \theta_{p-1} j} B_{j \varphi_2 \dots \varphi_q} \| \omega_1 \dots \omega_{p-1} \Phi_2 \dots \Phi_q \right\|, \quad (7.10)$$

and the theorem is proved.

The *mth compound* of a determinant $|A|$ is the cosignant determinant of the matrix composed of the minors of $|A|$ of order m arranged in the normal order.

The mth compound of a Cayley product of two determinants is equal, element to element, to the like Cayley product of the mth compounds of the determinants. (7.11)

Proof. Using the above matrices, let $|A|$ and $|B|$ be any determinants in which α_p and β_1 are signant, let $|AB|$ be their Cayley product, and let $|(A)_m|$, $|(B)_m|$, and $|(AB)_m|$ be the respective mth compounds.

In $|(AB)_m|$ the element whose locant is $\Theta_1 \cdots \Theta_{p-1} \Phi_2 \cdots \Phi_q$ is

$$|\sum_{i=1}^{n} A_{\theta_1 \dots \theta_{p-1} i} B_{i \varphi_2 \dots \varphi_q}|, \quad (7.12)$$

while in $|(A)_m| \cdot |(B)_m|$ the corresponding element is

$$\sum_{I=1}^{\binom{n}{m}} |\sum\nolimits_i A_{\theta_1 \dots \theta_{p-1} i} B_{i \varphi_2 \dots \varphi_q}|. \quad (7.13)$$

(The step leading above to (7.10) cannot here be taken, since it is a matrix and not a determinant step.)

These corresponding elements are equal by virtue of the Binet-Cauchy relation, the matrix of (7.12) being the product of the elongated matrices

$$\| A_{\theta_1 \dots \theta_{p-1} \alpha_p} \|, \qquad \| B_{\beta_1 \varphi_2 \dots \varphi_q} \|, \quad (7.14)$$

i.e., being the "C" of (6.3), while the $\binom{n}{m}$ terms of (7.13) are the products of corresponding determinants of the elongated matrices.

Adjoint.—The *adjoint* of a determinant is obtained by replacing each element by its cofactor; it can therefore be formed from the $(n-1)$th compound by reversing the order of the layers of each direction and multiplying by -1 each even-numbered layer of each signant direction; and it is equal in value to the $(n-1)$th compound because there are an even number of signant directions.

The adjoint of a Cayley product of two determinants is equal, element to element, to the like Cayley product of the adjoints of the determinants. (7.15)

Proof. Let $m = n - 1$ in (7.11). If the order of the layers in each direction in the compounds be reversed, the equality of corresponding elements still exists, since the choice of a " normal order " was arbitrary. It remains only to show that the equality is not impaired by multiplying by -1 each even-numbered layer of each signant direction in the modified compounds. Such multiplication with respect to signant directions in $\Theta_1 \cdots \Theta_{p-1}\Phi_2 \cdots \Phi_q$ will affect alike the two general elements in question. With respect to Θ_p and Φ_1, inasmuch as their values are everywhere *paired* in the product of the modified compounds, the resulting sign is $(-1)(-1)$ and has no effect.

8. The Scott Product

The product of two determinants $|A|$ and $|B|$ of classes p and q and order n can be expressed as a determinant $|S|$ of class $p + q - 1$, and order n, each of whose elements is the product of an element of $|A|$ and an element of $|B|$.

For convenience, fix on the last direction of $|A|$ and the first direction of $|B|$: either direction may be signant or nonsignant (unlike the directions combined in a Cayley product). Multiply each element of the ith layer of the last direction of the matrix A into each element of the ith layer of the first direction of B to form the ith layer of the pth direction of S:

$$S = \| A_{\alpha_1 \ldots \alpha_{p-1} \underset{\sim}{\alpha_p}} B_{\underset{\sim}{\beta_1} \beta_2 \ldots \beta_q} \|, \tag{8.1}$$

where a symbol of *linkage* joins α_p and β_1 into a bipartite index taking the n values 11, 22, $\cdots$, nn.

Each transversal of S is the product of a transversal of A and a transversal of B, every possible product appearing once and only once. (8.2)

Let $|S|$ be that determinant of S in which the signancy of the α's and β's is continued, so that $\alpha_p\beta_1$ has the signancy which is the result of the superposed signancies of α_p and β_1 and is therefore nonsignant if both α_p and β_1 are nonsignant, signant if α_p *or* β_1 is signant, and again nonsignant if α_p *and* β_1 are signant:

$$|S| = |A_{\alpha_1 \cdots \alpha_{p-1}\overbrace{\alpha_p}}^{(\pm)_x} B_{\overbrace{\beta_1\beta_2 \cdots \beta_q}}^{(\pm)_y}|, \tag{8.3}$$

x and y being the signancies of $|A|$ and $|B|$.

The terms of $|S|$ are the products of the terms of $|A|$ and $|B|$, hence

$$|S| = |A| \cdot |B|. \tag{8.4}$$

Evidently a Scott product of a determinant and a permanent, or of two permanents, may be formed.

A Scott product of any number of determinants may be formed by first forming a Scott product of two of them, then a Scott product of this and a third, and so on. At any stage, any index of the new factor may be linked with any index of the partial product to form a multipartite index taking the n values $1 \cdots 1$, $2 \cdots 2$, $\cdots$, $n \cdots n$, the signancy of that index in the new factor being superposed on its simple or complex signancy in the partial product.

To any minor of order m of a Scott product there *correspond* minors of order m, one in each factor of the product, obtained by restricting each index to the same m values that it has in the minor of the product.

Any minor of a Scott product of any number of determinants is a like Scott product of the corresponding minors of the determinants. (8.5)

9. Compounds of Scott Products

The mth compound of a Scott product of any number of determinants is a like Scott product of the mth compounds of the determinants. (9.1)

Proof. The general proof will be clear (in this and subsequent theorems) from that for two determinants. Their compounds are

$$\left| |A_{\theta_1 \ldots \theta_p}|_{\Theta_1 \ldots \Theta_p} \right|, \qquad \left| |B_{\varphi_1 \ldots \varphi_q}|_{\Phi_1 \ldots \Phi_q} \right|, \tag{9.2}$$

and the compound of their product is

$$\left| |A_{\theta_1 \ldots \theta_{p-1} \underset{\frown}{\theta_p} B_{\varphi_1 \varphi_2 \ldots \varphi_q}}|_{\Theta_1 \ldots \Theta_{p-1} \underset{\frown}{\Theta_p} \Phi_1 \Phi_2 \ldots \Phi_q} \right|; \tag{9.3}$$

and (9.3) is the Scott product of the compounds in (9.2).

The mth compound of a Scott product $|T|$ *of any number of 2-way determinants of the nth order is equal to* $|T|^{\binom{n-1}{m-1}}$. (9.4)

For, by (9.1) and the theory of 2-way determinants,

$$|(T)_m| = |(a)_m| \cdot |(b)_m| = |a|^{\binom{n-1}{m-1}} \cdot |b|^{\binom{n-1}{m-1}}.$$

The *conformed* $(n - m)$th compound of $|A|$ is the determinant obtained by replacing each element of the mth compound of $|A|$ by its algebraic complement in $|A|$.

The conformed $(n - m)$*th compound is equal in value to the* $(n - m)$*th compound.* (9.5)

For, the former may be got from the latter by reversing the order of the layers of each direction and multiplying the same layers of each signant direction by -1. The adjoint is an example.

The conformed $(n - m)$*th compound of a Scott product of any number of determinants is a like Scott product of the conformed* $(n - m)$*th compounds of the determinants.* (9.6)

Proof. Denote by Θ'_i the selection of $n - m$ values complementary to the m values in Θ_i, and by θ'_i those $n - m$ values individually. Denote by $\epsilon_{\Theta_i\Theta'_i}$ the sign of the α_i-range when the m values in Θ_i, in normal order, are followed by the $n - m$ values in Θ'_i. in normal order; and by $\epsilon^{(x)}_{\Theta\Theta'}$ the product of the signs of the signant ranges of $|A|$ when in each range the values are so ordered. Thus $\epsilon^{(x)}_{\Theta\Theta'}$ is the sign borne by the element of the conformed $(n - m)$th compound of $|A|$ whose locant is $\Theta'_1\Theta'_2 \cdots \Theta'_p$. Similarly for Φ and φ. The conformed $(n - m)$th compounds of $|A|$ and $|B|$ are then

$$\left|\epsilon^{(x)}_{\Theta\Theta'}|A_{\theta'_1\ldots\theta'_p}|_{\Theta'_1\ldots\Theta'_p}\right|, \qquad \left|\epsilon^{(y)}_{\Phi\Phi'}|B_{\varphi'_1\ldots\varphi'_q}|_{\Phi'_1\ldots\Phi'_q}\right|, \tag{9.7}$$

and the conformed $(n - m)$th compound of the Scott product of $|A|$ and $|B|$ is

$$\left|\epsilon^{(x)}_{\Theta\Theta'}\epsilon^{(y)}_{\Phi\Phi'}|A_{\theta'_1\ldots\theta'_{p-1}\underbrace{\theta'_p}B_{\varphi'_1\varphi'_2\ldots\varphi'_q}}|_{\theta'_1\ldots\theta'_{p-1}\underbrace{\theta'_p\Phi'_1\Phi'_2\ldots\Phi'_q}}\right|; \tag{9.8}$$

and this is also the Scott product of the conformed compounds in (9.7).

Any minor of order r of the conformed $(n - m)$th compound of a Scott product $|T|$ of any number of 2-way determinants of order n, is equal to the algebraic complement of the corresponding minor of the mth compound multiplied by $|T|^{r-\binom{n-1}{m-1}}$. (9.9)

Proof. Determinants will here be denoted by Gothic letters without bars. The Scott products are all of the same type. Let T be a Scott product of the 2-way determinants a and b, let T_m, a_m, b_m be their mth compounds, and let $\mathsf{T}_{\bar{m}}$, $\mathsf{a}_{\bar{m}}$, $\mathsf{b}_{\bar{m}}$ be the conformed $(n - m)$th compounds. Let ${}_r\mathsf{T}_{\bar{m}}$ be the given minor of $\mathsf{T}_{\bar{m}}$ of order r, and $\epsilon_{r,\bar{r}}\mathsf{T}_m$ be the specified algebraic complement. Let $r - \binom{n-1}{m-1} = \rho$. We are to show that

$${}_r\mathsf{T}_{\bar{m}} = \epsilon_{r,\bar{r}}\mathsf{T}_m \cdot \mathsf{T}^{\rho}. \tag{9.10}$$

By (9.6) $\mathsf{T}_{\bar{m}}$ is a Scott product of $\mathsf{a}_{\bar{m}}$ and $\mathsf{b}_{\bar{m}}$. Hence if ${}_r\mathsf{a}_{\bar{m}}$

and $_r\mathbf{b}_{\overline{m}}$ be the minors corresponding to $_r\mathbf{T}_{\overline{m}}$, the latter is their Scott product, by (8.5). Again, for the same reason, $_r\mathbf{T}_m$ is the Scott product of $_r\mathbf{a}_m$ and $_r\mathbf{b}_m$, and as ϵ_r is expressible as the product of the signs ϵ_{r_a} and ϵ_{r_b} required to make the 2-way minors into algebraic complements, (9.10) becomes:

$$_r\mathbf{a}_{\overline{m}} \cdot {}_r\mathbf{b}_{\overline{m}} = \epsilon_{r_a} \cdot {}_r\mathbf{a}_m \cdot \epsilon_{r_b} \cdot {}_r\mathbf{b}_m \cdot \mathbf{a}^e\mathbf{b}^e. \quad (9.11)$$

But from the theory of 2-way determinants,

$$_r\mathbf{a}_{\overline{m}} = \epsilon_{r_a} \cdot {}_r\mathbf{a}_m \cdot \mathbf{a}^e, \qquad {}_r\mathbf{b}_{\overline{m}} = \epsilon_{r_b} \cdot {}_r\mathbf{b}_m \cdot \mathbf{b}^e, \qquad (9.12)$$

and the theorem is established.

For the special case of the adjoint,

Any minor of order r of the adjoint of a Scott product $|T|$ *of any number of 2-way determinants of order n, is equal to the algebraic complement of the corresponding minor of* $|T|$, *multiplied by* $|T|^{r-1}$. (9.13)

A similar theorem holds if no ϵ's are used, so that the elements of the adjoint are deprived of their signs, the algebraic complement mentioned being at the same time reduced to a plain complement, viz.:

Any minor of order r of the layer-reversed $(n-1)$*th compound of a Scott product* $|T|$ *of any number of 2-way determinants of order n, is equal to the complementary of the corresponding minor of* $|T|$, *multiplied by* $|T|^{r-1}$. (9.14)

The Law of Complementaries and the Law of Extensible Minors in the theory of 2-way determinants, here take the following forms, by virtue of the preceding theorems:

Law of Complementaries.—To every homogeneous relation between a number of the minors of a Scott product $|T|$ *of any number of 2-way determinants, the relation being general to that type of Scott product, there corresponds another relation derivable from the former by merely substituting for every minor its cofactor (or, alternatively, for every minor its complement) in* $|T|$ *and then multiplying any*

term by such a power of $|T|$ *as will make the terms of the same degree.* (9.15)

Law of Extensible Minors.—To every homogeneous relation between a number of the minors of a Scott product $|T|$ *of any number of 2-way determinants of order n, the relation being general to that type of Scott product, there corresponds another relation between minors of any like Scott product* $|T_h|$ *of order* $n + h$ *of which* $|T|$ *is a minor, derivable from the former relation by merely substituting for every minor* $|R|$ *of order r that minor of order* $r + h$ *of* $|T_h|$ *which includes* $|R|$ *and the complement* $|\overline{T}|$ *of* $|T|$*, and then multiplying any term by such a power of* $|\overline{T}|$ *as will make the terms of the same degree.* (9.15)

10. References, etc.

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In the last two references to M. Lecat, higher determinants, including full-sign determinants, are called *riciens*. Full-sign determinants were studied first (and very carefully) by Cayley, in 1843. Later writers added much to the subject, though most of them failed to observe a restriction noted by Cayley—that his law of multiplication was inapplicable if both factors were determinants of odd class. The introduction of less-than-full-sign determinants by the writer in 1918 removed the restriction and broadened and simplified the theory. There is every reason for calling the more inclusive class of functions *determinants*, the name accepted and used hitherto by Lecat, and used always by Hitchcock and the writer. It is inappropriate to call all determinants by his name. Besides, as with *jacobian*, *hessian*, etc., a determinant whose elements are formed in a particular manner is suggested, and much confusion would result.

In general it seems to the writer a matter of the utmost importance that words and symbols familiar in the theory of ordinary determinants be so far as possible used for higher determinants.

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MATHEMATICAL THEORY OF THE PROCESS OF CONSOLIDATION OF MUD DEPOSITS

By Alberto Ortenblad

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I. Introduction

The most important mud deposits are formed at the mouths of large rivers. Here the decrease in the velocity of their streams, and the presence of certain salts in the ocean water, which act as electrolytes when the grains are of colloidal size, cause the sedimentation of the suspended matter. In time deltas are formed there. Other important deposits occur on the beds and sides of natural streams and lakes. This process of sedimentation goes on continuously, and large areas of land take the place of previous water surfaces. As time goes on, these mud deposits, besides increasing in size, become more or less consolidated and often serve as a foundation for large and important cities.

Therefore one can easily see the importance of a careful study of the behavior of such mud deposits under varying conditions. By local observations and by a single laboratory experiment, which will be described later, we can determine the rate at which sedimentation takes place, whether it is uniform or not, the specific weight, and other physical characteristics of the deposit: the coefficient of permeability, etc. Having these data, we are

in a position to fully determine and predict the state of stress, water-content, and settlement at any time.

The knowledge of the stress conditions in a mud deposit is important in so far as it explains practically all phenomena connected with foundation construction on soft grounds.

As to the structure of the sediment, two cases will be considered, namely: that of a sediment of homogeneous material and that in which there exists within the sediment a thin layer of less permeable material.

Three aspects of the process of consolidation of such mud deposits will be considered. (*a*) Consolidation due to the weight of the material itself; (*b*) consolidation due to the weight of a top fill of very permeable material or its equivalent—evaporation; and (*c*) consolidation by drainage. In all cases the bottom surface of the sediment will be assumed horizontal and impermeable.

The theory developed will hold not only for mud deposits but also for clays and fine-grained materials in general, provided that no air is present in the voids of the material.

Free use will be made of Fourier's Series and Integrals in the attempts to solve the differential equations. The application of Heaviside's Operational Method of solving differential equations will be illustrated in one case.

In conclusion, the writer wishes to thank Dr. Charles Terzaghi for his valuable assistance in connection with this thesis, as well as the members of the staff of the Civil Engineering Department with whom he came in contact, and takes this opportunity to express his appreciation.

II. General Laws and Definitions

1. Definitions. *Volume of Voids* (n) is the ratio between the total volume of voids (N) and the total volume of material (V):

$$n = \frac{N}{V}.$$

Voids Ratio (ϵ) is the ratio between the total volume of voids and the total volume of solid matter (V_s):

$$\epsilon = \frac{N}{V_s} = \frac{N}{V - N} = \frac{n}{1 - n}.$$

The voids-ratio then, is a measure of the water content per unit of solid matter.

Granular Pressure or Stress (p) is the intensity of the pressure acting between the grains of the material at a given point and in a specified direction.

Hydrodynamic Pressure or Stress (w) is the excess of the intensity of the water pressure over the hydrostatic pressure at a given point, and acts with the same intensity in all directions.

Reduced Dimensions (x, y, z) are dimensions (or distances) equivalent to a volume of voids (or voids-ratio) equal to zero. We are forced to use reduced dimensions because if there is a change in either or both of the above defined pressures, in a given mass of mud, there will be, as a consequence, a change in its water content which in turn will cause a change in its dimensions. Hence the true dimensions are as variable as the internal stresses themselves, while the reduced dimensions are unaltered by any one of the above changes.

2. Darcy's Law. This law states that if water is admitted through a layer of granular material of cross-sectional area (A) and thickness l, then the quantity of water Q, percolating through any section of the layer, perpendicular to its cross section, per unit of time, is given by

$$Q = ikA. \tag{1}$$

In this formula, i is the hydraulic gradient which, for steady flow of water, is equal to the ratio h/l (Fig. 1), h being the hydraulic head and k the coefficient of permeability of the material. From the above formula we see that the coefficient of permeability is equal to the velocity of percolation per unit of time, under a hydraulic head equal to the thickness of the layer, i.e., under a

hydraulic gradient equal to unity. If, however, the flow is not steady, and if we call w the hydraulic head at a section distant s from any convenient reference line, then

$$i = \frac{\partial w}{\partial s}$$

at the section under consideration.

Hence,

$$Q = K \frac{\partial w}{\partial s} A. \tag{2}$$

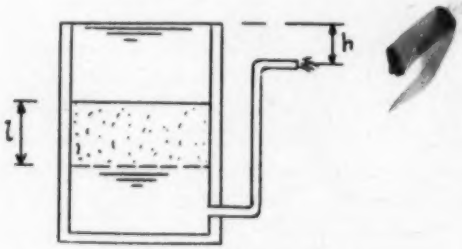

FIG. 1.

This law is strictly true for laminary flow of water such as is generally that through fine-grained materials. It has been verified by many experiments.[1]

3. Laboratory Experiment. With this experiment we aim to obtain two curves showing the variation of water-content with pressure and the variation in the coefficient of permeability with water content. The curves representing these variations are shown in Fig. 2.

The apparatus used to obtain the $p - \epsilon$ curve consists of a container in which a layer of the material to be tested is placed and then covered by filter paper and sand immersed in water. The pressure p is then applied at the top and varied through the

[1] For formulae for the coefficient of permeability, k, for sands and clays and for a discussion regarding the validity of the law of Darcy, see: "Principles of Soil Mechanics," by Dr. Charles Terzaghi, *Engineering News Record*, Nov. 14, 1925. Cf. also Terzaghi's "Erdbaumechanik," Chapter IV.

range desired.[2] The pressure-moisture curve thus obtained is valid for the case of linear flow of water. If, however, there exists a flow of water in more than one direction, an apparatus similar to the above one can be made having lateral filters,

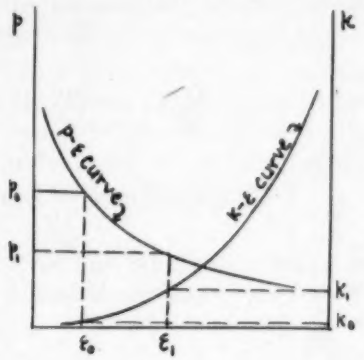

FIG. 2.

provided the lateral pressures are known functions of the top pressure p. The present discussion will be limited to the one dimensional case.

The relation between the water-content (ϵ) and the pressure p is given by the following equation [2]

$$\epsilon = -\alpha' \log_e (p + p_1) - \beta'(p + p_1) + C,$$

where α', β', p_1 and C are constants. Since β' is very small [3] the above equation may be written

$$\epsilon = -\alpha \log_e (p + \beta) + C,$$

and

$$\frac{d\epsilon}{dp} = -\frac{\alpha}{p + \beta}, \tag{3}$$

which is the equation of the tangent to the curve.

[2] Cf. "Principles of Soil Mechanics," by C. Terzaghi, *Engineering News Record*, Nov. 19, 1925 and Nov. 26, 1925. Also Terzaghi's "Erdbaumechanik," Fig. 13, p. 83 and p. 87, equation (24).

[3] "Erdbaumechanik," p. 162.

The ratio $a = -de/dp$ = modulus of compression, may be taken as constant and equal to the average of its extreme values if the range through which p varies is not very large.[4]

The ratio k/a (where k is the coefficient of permeability) was found to be almost constant for materials with plastic consistency,[5] and will be so considered in what follows. Hence we can write:

$$a = -\frac{de}{dp} = \frac{\alpha}{p+\beta}, \tag{4}$$

and

$$\frac{k}{a} = c = \text{constant}. \tag{5}$$

The constants α and β should be determined for the range through which the pressure p ranges in the actual problem dealt

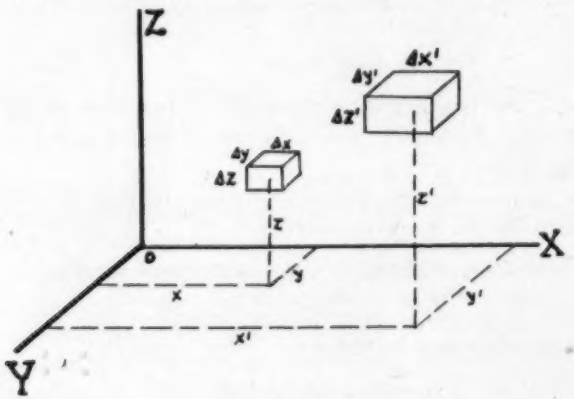

FIG. 3.

with. For very small pressures, β is very large, while for large values of p it is very small.

[4] C. Terzaghi: *Engineering News Record*, Nov. 26, 1926, p. 874. Also "Erdbaumechanik," p. 141.

[5] "Erdbaumechanik," Fig. 22, p. 121. Also pp. 126–127.

4. General Equation of Stresses in a Mass of Granular Material with Water-Filled Voids. Let a mass of mud be referred to a system of rectangular coordinates. Let O be its origin and let (x', y', z') be the coordinates of the center of an elementary prism whose sides are $\Delta x'$, $\Delta y'$, $\Delta z'$. Assume the flow of water to be in the positive directions of the coordinate axes X, Y and Z, and let w' be the hydrodynamic pressure at the center of this elementary prism; i.e., at point (x', y', z').

The hydrodynamic pressure intensities at the faces of this prism—which is assumed to be very small and finally approach the limiting value zero—will therefore be:

$$\text{Plane } YZ\text{: left face: } w' - \frac{1}{2}\frac{\partial w'}{\partial x'}\Delta x';$$

$$\text{right face: } w' + \frac{1}{2}\frac{\partial w'}{\partial x'}\Delta x'.$$

$$\text{Plane } XZ\text{: front face: } w' - \frac{1}{2}\frac{\partial w'}{\partial y'}\Delta y';$$

$$\text{back face: } w' + \frac{1}{2}\frac{\partial w'}{\partial y'}\Delta y'.$$

$$\text{Plane } XY\text{: lower face: } w' - \frac{1}{2}\frac{\partial w'}{\partial z'}\Delta z';$$

$$\text{upper face: } w' + \frac{1}{2}\frac{\partial w'}{\partial z'}\Delta z'.$$

Now, according to the law of Darcy, the quantity of water (Q), percolating normally through a plane surface whose sectional area is A, per unit of time, in a direction s, is given by

$$Q = k\frac{\partial w'}{\partial s}A$$

and since the flow of water is in the positive direction of s, w' must decrease as s increases and therefore $\partial w'/\partial s$ is a negative quantity.

Therefore, the time rate of percolation is given numerically by

$$Q = -k\frac{\partial w'}{\partial s}A.$$

Applying this relation to parallel opposite faces of the elementary prism, we get the following values of Q for the faces parallel to the YZ plane:

$$\text{left face:} \quad -k\Delta y'\Delta z'\frac{\partial}{\partial x'}\left(w' - \frac{1}{2}\frac{\partial w'}{\partial x'}\Delta x'\right)$$

$$= -k\Delta y'\Delta z'\left(\frac{\partial w'}{\partial x'} - \frac{1}{2}\frac{\partial^2 w'}{\partial x'^2}\Delta x'\right),$$

$$\text{right face:} \quad -k\Delta y'\Delta z'\frac{\partial}{\partial x'}\left(w' + \frac{1}{2}\frac{\partial w'}{\partial x'}\Delta x'\right)$$

$$= -k\Delta y'\Delta z\left(\frac{\partial w'}{\partial x'} + \frac{1}{2}\frac{\partial^2 w'}{\partial x'^2}\Delta x'\right).$$

Subtracting the second expression from the first, we find the difference between inflow and outflow per unit of time to be

$$k\Delta x'\Delta y'\Delta z'\frac{\partial^2 w'}{\partial x'^2};$$

and similar expressions for the other pairs of parallel faces.

Therefore the difference between the total inflow and the total outflow of water in the elementary prism per unit time is

$$k\Delta x'\Delta y'\Delta z'\left(\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2}\right). \tag{6}$$

This difference should be equal to the time rate of change (an increase in this case) in water content of the elementary prism.

At this point, attention is called to the fact that, in the above differentiations, k has been considered as a constant, and equal to the average value of the coefficient of permeability. Thus, to a continuous change in the granular pressure at a given point from p_0 to p_1 say, there is a corresponding change in water content

per unit of solid matter of $\epsilon_1 - \epsilon_0$, and therefore the coefficient of permeability varies continuously from a value k_0 to a value k_1. If the changes in pressures are relatively small, we are justified in assuming k constant, and having a value intermediate between k_0 and k_1.

If, however, due account is to be taken of the variation in the values of k, we should proceed as follows: Let k be the value of the coefficient of permeability at the center of the elementary prism; then its value at the two faces parallel to the YZ plane are

$$k - \frac{1}{2}\frac{\partial k}{\partial x'}\Delta x' \qquad \text{and} \qquad k + \frac{1}{2}\frac{\partial k}{\partial x'}\Delta x'.$$

Hence the time rate of percolation at these two faces is given by

$$-\left(k - \frac{1}{2}\frac{\partial k}{\partial x'}\Delta x'\right)\frac{\partial}{\partial x'}\left(w' - \frac{1}{2}\frac{\partial w'}{\partial x'}\Delta x'\right)\Delta y'\Delta z',$$

and

$$-\left(k + \frac{1}{2}\frac{\partial k}{\partial x'}\Delta x'\right)\frac{\partial}{\partial x'}\left(w' + \frac{1}{2}\frac{\partial w'}{\partial x'}\Delta x'\right)\Delta y'\Delta z'.$$

Differentiating and subtracting the second expression from the first we find

$$\Delta x'\Delta y'\Delta z'\left(k\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial k}{\partial x'}\frac{\partial w'}{\partial x'}\right),$$

as the difference between the inflow and outflow of water per unit of time. Therefore the difference between total inflow and outflow per unit of time in the prism is expressed by

$$\Delta x'\Delta y'\Delta z'\left[k\left(\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2}\right) + \frac{\partial k}{\partial x'}\frac{\partial w'}{\partial x'} + \frac{\partial k}{\partial y'}\frac{\partial w'}{\partial y'} + \frac{\partial k}{\partial z'}\frac{\partial w'}{\partial z'}\right]. \quad (7)$$

In order to find an expression for the time rate of change in water content, we have to introduce reduced dimensions. Let the new (reduced) dimensions of our elementary prism be Δx, Δy and Δz and let (x, y, z) be the coordinates (reduced) of its

center, where the hydrodynamic pressure is now w instead of w'. These transformations do not change the value of w numerically, for it is still equal to w' but referred to a new system of coordinates. w is then equivalent to the temperature (or potential) difference in the case of flow of heat (or electricity) through an isotropic body.

The expressions (6) and (7) for the time rate of change in water content, in terms of the reduced dimensions are:

$$k\Delta x\Delta y\Delta z\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{8}$$

and

$$\Delta x\Delta y\Delta z k\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\partial k}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial k}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial k}{\partial z}\frac{\partial w}{\partial z}. \tag{9}$$

Now we have seen that ϵ measures the water content per unit volume of solid matter, and if we let $\Delta\epsilon$ be the change in water content per unit of volume of solid matter in the time element Δt, then the change in water content of the elementary prism $\Delta x\Delta y\Delta z$ per unit of time will be

$$\Delta x\Delta y\Delta z\frac{\partial \epsilon}{\partial t}. \tag{10}$$

But

$$\frac{\partial \epsilon}{\partial t}=\frac{\partial \epsilon}{\partial p}\frac{\partial p}{\partial t}=-a\frac{\partial p}{\partial t},$$

Therefore (10) becomes

$$-\Delta x\Delta y\Delta z a\frac{\partial p}{\partial t}. \tag{11}$$

Equating (8) and (9) to (11) we obtain:

$$\frac{\partial p}{\partial t}=-\frac{k}{a}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{12}$$

if k is taken as constant, and

$$\frac{\partial p}{\partial t}=-\frac{k}{a}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{1}{a}\left(\frac{\partial k}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial k}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial k}{\partial z}\frac{\partial w}{\partial z}\right), \tag{13}$$

if due account is taken of the variation in the coefficient of permeability.

These are the fundamental differential equations for the stress distribution in a mass of granular material.

Equation (12) is similar to Fourier's equation of heat conduction, the only difference being in having p instead of w in the left-hand member.

In applying the differential equation (either (12) or (13)) to mud deposits, two distinct stages must be considered. The first stage comprises the lapse of time between the sedimentation of the first and last layers of material, that is, when material is still being deposited. After this stage no more material is added to the deposit, and its consolidation takes place either under the weight of the solid matter itself or under the influence of an external load or evaporation. This is the second stage.

In the first stage. since the (reduced) dimensions of the deposit change continuously with the time, it is evident that t is not an independent variable, while in the second stage the time and space variables are completely independent of each other.

We will designate by t' the time in the first stage and by t that in the second stage.

III. Problem I. Mud Deposit of Homogeneous Material—Uniform Rate of Sedimentation—Bottom Surface Impermeable and Horizontal. Consolidation Due to the Weight of the Deposit Itself

1. **Formation of mud deposits.** Fig. 4 shows a deposit of homogeneous material of total thickness H. As the true thickness at any time t' varies from $H' = 0$ to a final value H, the reduced thickness varies from $h' = 0$ to $h' = h$. We choose the bottom surface as the origin of coordinates. Since the conditions in any horizontal plane are the same at all points, there will be flow of water in only one direction, and we can take a cylinder of the material of unit cross-sectional area whose height

increases continuously up to a value H (reduced = h) to represent the actual conditions.

Fig. 5 illustrates the case of sedimentation in an inclined plane. Here there will be flow of water in two directions, but if the slope of the plane is small, as is usually the case, the lateral flow may

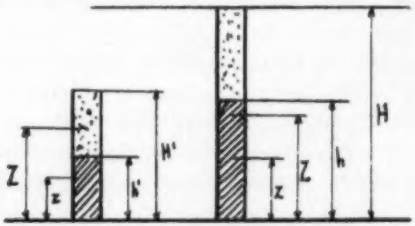

FIG. 4.

be disregarded and the problem treated under the case of linear flow of water.

Before attempting to solve the problem illustrated by Fig. 4, we will show why there is flow of water in the vertical direction and how it affects the state of internal stresses.

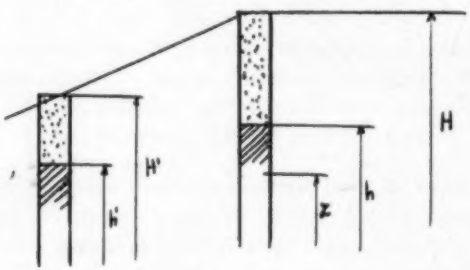

FIG. 5.

Let $\gamma + 1$ be the specific weight of the deposited material when dry, then γ will be its specific weight under water. Let g be the quantity (grams) of dry substance deposited per unit

surface per unit time, and q the corresponding quantity under water which is different from g since it is expressed in grams, i.e., as weight.

Then

$$q = \frac{\gamma}{\gamma + 1} g. \tag{14}$$

Consider the deposit at a time when its total reduced thickness is h' (Fig. 6). As time goes on, more and more material is being sedimented so that after a certain lapse of time the top surface of the deposit is located at a height h'' (reduced) from the bottom. Pick out an arbitrary point at a distance z (reduced) from the bottom. As the top surface of the deposit increased from a position a to a position b the total weight of the material (or the

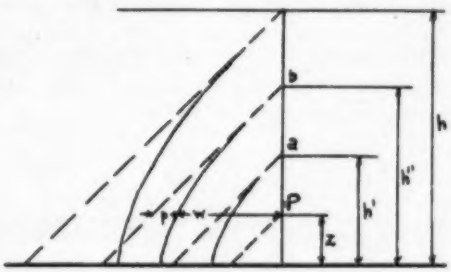

FIG. 6.

total pressure) at P increased from $\gamma(h' - z)$ to $\gamma(h'' - z)$. Part of this increase in the total pressure at P is taken up by the solid particles, and part by the water. Therefore the hydrodynamic pressure increases continuously. But since a difference in hydrodynamic pressure implies flow of water (in the same way as a difference in temperature (or potential) implies flow of heat (or electricity)), we will have water coming out continuously from the top surface. As a consequence the water content at P decreases as the thickness of the deposit increases, and the smaller z is the smaller will be the water content for a given thickness of

the deposit. On the other hand, the granular pressure (p) at P must increase with the increase in the thickness of the deposit (as can be seen from pressure-moisture curves in Section II), and also the smaller z is for a given thickness, the greater will be the granular pressure p.

We will now pass to the solution of our problem, first by applying the differential equation for k constant and then for k variable.

The differential equations (12) and (13) for the case of linear flow of water become:

$$\frac{\partial p}{\partial t} = -\frac{k}{a}\frac{\partial^2 w}{\partial z^2} = -c\frac{\partial^2 w}{\partial z^2}, \tag{15}$$

and

$$\frac{\partial p}{\partial t} = -c\frac{\partial^2 w}{\partial z^2} - \frac{1}{a}\frac{\partial k}{\partial z}\frac{\partial w}{\partial z}. \tag{16}$$

2. Solution of Equation (15). (a) *First stage.* Since time is not an independent variable during the first stage of the process of consolidation, we write (15) as follows:

$$\frac{\partial p}{\partial t'} = -c\frac{\partial^2 w}{\partial z^2}. \tag{15a}$$

Let the total reduced thickness of the deposit be h' and let t'_i be

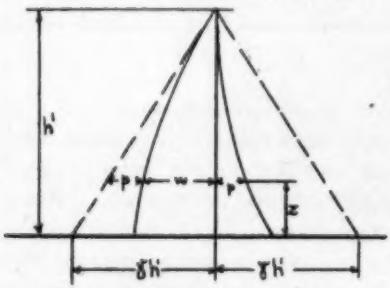

FIG. 7.

the time required for the deposit to reach a height h'. The volume of material sedimented per unit area and unit of time is

q/γ and this is also the thickness of the layer of material deposited per unit of time. Therefore in a time interval t'_1, the thickness of material deposited is $(q/\gamma)t'_1$, and this must be equal to h'.

$$t_1' = \frac{\gamma}{q} h'.$$

Consider now any point at a height z from the bottom such that $z \leq h'$.

The time required for the layer at height z to come into existence is

$$t_z' = \frac{\gamma}{q} z.$$

Of course no flow of water through the layer located at a distance z from the bottom could possibly exist before the layer itself had come into existence. Since the variable time in the differential equation (15a) concerns the interval from which water begins to percolate through the layer under consideration (i.e., from which the layer has come into existence) up to any value t', we must have $t' = 0$ for a time $t_z = (\gamma/q)z$, and therefore

$$t' = \frac{\gamma}{q}(h' - z). \tag{17}$$

The total downward pressure acting at a height z from the bottom is $\gamma(h' - z)$, and this must be balanced by the granular and hydrodynamic pressures in order to maintain equilibrium; hence

$$p + w = \gamma(h - z), \tag{18}$$

or

$$\frac{\partial w}{\partial z} = -\gamma - \frac{\partial p}{\partial z}, \tag{19}$$

and

$$\frac{\partial^2 w}{\partial z^2} = -\frac{\partial^2 p}{\partial z^2}. \tag{20}$$

From (17) and (19) we obtain the relation

$$\frac{\partial p}{\partial t'} = -\frac{q}{\gamma}\frac{\partial p}{\partial z}.$$

By substitution in (15a) we get

$$-\frac{q}{\gamma}\frac{dp}{dz} = c\frac{d^2p}{dz^2},$$

or

$$\frac{d^2p}{dz^2} + b\frac{dp}{dz} = 0, \tag{21}$$

where

$$b = \frac{q}{\gamma c}. \tag{22}$$

The solution of (21) is

$$p = A + Be^{-bz}, \tag{23}$$

where A and B are constants of integration.

To determine A and B we know that p and w are zero for $z = h'$ (Fig. 7) and since the bottom surface is assumed to be impermeable, we must have $dw/dz = 0$ for $z = 0$, that is, the curve of hydrodynamic pressure must be perpendicular to the bottom surface. To $dw/dz = 0$ corresponds $dp/dz = -\gamma$ and therefore the boundary conditions are

$$z = h' : p = 0, \qquad w = 0, \tag{24}$$

$$z = 0 : \frac{dw}{dz} = 0, \qquad \frac{dp}{dz} = -\gamma. \tag{25}$$

From conditions (24) and (25) we find

$$A = \frac{\gamma}{b} \qquad \text{and} \qquad B = -\frac{\gamma}{b}e^{-bh'}.$$

Therefore

$$p = \frac{\gamma}{b}(e^{-bz} - e^{-bh'}). \tag{26}$$

The hydrodynamic pressure is then found from (18). The pressure distribution when $h' = h$, i.e., at the time when sedimentation is assumed to stop is

$$p = \frac{\gamma}{b}(e^{-bz} - e^{-bh}), \tag{27}$$

and

$$w = \gamma(h - z) - \frac{\gamma}{b}(e^{-bz} - e^{-bh}). \tag{28}$$

(b) *Second Stage.* We will consider here the effect of the weight itself of the material on the process of consolidation. In this stage the time is independent of z and therefore from (18) we get

$$\frac{\partial p}{\partial t} = -\frac{\partial w}{\partial t}.$$

Substituting in (15) we have the following differential equation:

$$\frac{\partial w}{\partial t} = c\frac{\partial^2 w}{\partial z^2}, \tag{29}$$

which is identical with Fourier's equation for the linear flow of heat, provided w and c are made to correspond to the temperature difference and the diffusivity respectively.[6]

If we make use of this analogy, our problem will be equivalent to that of non-steady flow of heat through a plate of isotropic material of thickness h, having one bounding plane impermeable to heat and the other at zero temperature.

We will solve (29) by Fourier's method;[7] we will find a Fourier series development for w which will satisfy (29) and also the boundary conditions.

The property that the bottom surface ($z = 0$) is impermeable gives rise to the following interpretation.[8] Suppose that we have a plate of thickness equal to $2h$ (Fig. 8), provided with a plane of separation at its center which is impermeable to heat. Let both halves of this plate have identical temperature distri-

[6] For a thermodynamic analogy of this problem, see "Principles of Soil Mechanics," by C. Terzaghi, *Engineering News Record*, Nov. 26, 1925, and also Terzaghi's "Erdbaumechanic," pp. 142–143.

[7] Cf. Byerly's "Fourier's Series and Spherical Harmonics," and also Ingersoll and Zobel's "Mathematical Theory of Heat Conduction," pp. 44–65.

[8] Cf. Ingersoll-Zobel's "Mathematical Theory of Heat Conduction," p. 72 and p. 107.

butions as shown in Fig. 8. Then either half (1) or (2) may be removed without changing the temperature distribution on the other half, since there is no flow of heat through the central plane. On making the transformation

$$z = h - x, \tag{30}$$

the boundary conditions will be

$$w = 0 \quad \text{for} \quad x = 0,$$

and

$$w = 0 \quad \text{for} \quad x = 2h. \tag{31}$$

In order to simplify our equations we will count the time so

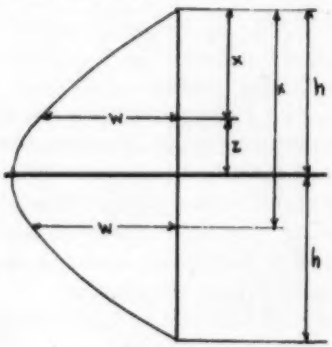

FIG. 8.

that $t = 0$ corresponds to the time when sedimentation no longer occurs.

The initial distribution of the hydrodynamic pressure (or temperature) throughout the layer of thickness $2h$ is, according to equations (18) and (26)

$$w_1 = f_1(x) = \gamma x - \frac{\gamma}{b} e^{-bh}(e^{bz} - 1)$$

for $t = 0$ and $0 \leq x \leq h$, (32)

$$w_2 = f_2(x) = \gamma(2h - x) - \frac{\gamma}{b} e^{-bh}\left[e^{b(2h-x)} - 1\right]$$

for $t = 0$ and $h \leq x \leq 2h$. (33)

Equation (29) after the transformation (30), becomes:

$$\frac{\partial w}{\partial t} = c\frac{\partial^2 w}{\partial x^2}. \tag{34}$$

To solve (34) let $w = e^{\alpha t + \beta x}$ where α and β are constants to be determined. Substituting this assumed value of w in (34) we obtain the relation

$$\alpha = c\beta^2$$

and in order to have our solution in terms of trigonometric functions (instead of hyperbolic) we set

$$\beta^2 = -\lambda^2 \quad \text{or} \quad \beta = \pm i\lambda \quad \text{where} \quad i = \sqrt{-1}.$$

Therefore $w = e^{-c\lambda^2 t \pm i\lambda x}$ is a solution of (34). Remembering that $e^{\pm iy} = \cos y \pm i \sin y$, we shall have

$$w = e^{-c\lambda^2 t} \cos \lambda x, \tag{35}$$

and

$$w = e^{-c\lambda^2 t} \sin \lambda x, \tag{36}$$

as particular (or partial) solutions of equation (34).

Now solution (36) will satisfy both of the boundary conditions (31) for all values of t provided that we take

$$\lambda = \frac{n\pi}{2h},$$

where n is an integer ($n = 1, 2, 3, \ldots$).

Therefore

$$w = e^{-\frac{cn^2\pi^2 t}{4h^2}} \sin \frac{n\pi}{2h} x \tag{37}$$

is a particular solution of (34).

If, in (37), we assign to n any arbitrary positive integral value, the equation thus obtained will satisfy both (31) and (34) and will therefore be a particular solution. Consequently there are an infinite number of particular solutions satisfying (31) and (34). Any of these particular solutions when multiplied by a

constant is also a solution, and therefore a solution of equation (34) is given by

$$w = \sum_{n=1}^{n=\infty} a_n e^{-\frac{cn^2\pi^2 t}{4h^2}} \sin \frac{n\pi}{2h} x, \tag{38}$$

where each a_n is a constant.

To complete the solution it remains only to determine a_n as a function of n. To do this we make use of the fact that the initial distribution of hydrodynamic pressure is known (equations (32) and (33)), that is, for $t = 0$, $w = \varphi(x)$ where $\varphi(x)$ stands for both $f_1(x)$ and $f_2(x)$.

Therefore

$$\varphi(x) = \sum_{n=1}^{n=\infty} a_n \sin \frac{n\pi}{2h} x.$$

From this expression we obtain

$$a_n = \frac{1}{h} \int_0^{2h} \varphi(x) \sin \frac{n\pi}{2h} x dx.$$

Substituting in this expression the two values of (x) given by (32) and (33) we get

$$a_n = \frac{\gamma}{h} \left\{ \int_0^h \left[x - \frac{e^{-bx}}{b} (e^{bx} - 1) \right] \sin \frac{n\pi}{2h} x dx + \int_h^{2h} \left\{ 2h - x - \frac{e^{-bh}}{b} \left[e^{b(2h-x)} - 1 \right] \right\} \sin \frac{n\pi}{2h} x dx \right\}.$$

Integrating and simplifying we obtain

$$\frac{h}{\gamma} a_n = -\frac{Ah^2}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} \right) + \frac{4h^2}{n^2\pi^2} \left(2 \sin \frac{n\pi}{2} - n\pi \cos \frac{n\pi}{2} + n\pi \cos n\pi \right) - \frac{2he^{-bh}}{n\pi b} (\cos n\pi - 1) - \left[\frac{e^{-bh}}{b} \right] \left[\frac{e^{bh} \left(2b \sin \frac{n\pi}{2} \right) - \frac{n\pi}{2h} (\cos n\pi - 1)}{b^2 + \frac{n^2\pi^2}{4h^2}} \right].$$

The right-hand member of this equation vanishes for all *even* values of n and therefore only *odd* values of n should be considered. When n is odd we have:

$$\frac{h}{\gamma}a_n = 8h^2\left[\frac{\sin\frac{n\pi}{2}}{n^2\pi^2} - \frac{\sin\frac{n\pi}{2} - \frac{2hb}{n\pi}e^{-bh}}{4h^2b^2 + n^2\pi^2}\right],$$

or

$$a_n = \frac{16\gamma h^2 b}{\pi}\cdot\frac{\frac{2hb}{n\pi}\sin\frac{n\pi}{2} + e^{-bh}}{n(4h^2b^2 + n^2\pi^2)}. \tag{39}$$

Therefore

$$w = \frac{16\gamma h^2 b}{\pi}\sum_{n=1,3,5,\ldots}\left[\frac{\frac{2hb}{n\pi}\sin\frac{n\pi}{2} + e^{-bh}}{n(4h^2b^2 + n^2\pi^2)}\right] \times\left[e^{-\frac{cn^2\pi^2}{4h^2}t}\sin\frac{n\pi}{2h}(h - z)\right], \tag{40}$$

is the equation giving the value of the hydrodynamic pressure for any positive values of t and z. Knowing w we can compute p from equation (18).

Equation (40) is seen to satisfy the differential equation (34) and both of the boundary conditions: $w = p = 0$ for $z = 0$ and $\partial w/\partial z = 0$ for $z = 0$ and for all values of t. When $t = \infty$ equation (40) gives $w = 0$, $p = \gamma(h - z)$ as should be the case.

The series represented by equation (40) converges very rapidly (as will be shown later in applying it to a concrete problem), so that an approximate solution may be obtained by setting $n = 1$, and disregarding the subsequent terms in the series. Doing this we have

$$w = \frac{16\gamma h^2 b}{\pi}\frac{\frac{2hb}{\pi} + e^{-bh}}{4h^2b^2 + \pi^2}e^{-\frac{c\pi^2}{4h^2}t}\sin\frac{\pi}{2h}(h - z). \tag{41}$$

We will now attempt a solution for the case in which due account is taken of the variation of the coefficient of permeability k, and the modulus of compression a.

3. Solution of Equation (16). (a) *First Stage.* From equation (3) we have

$$a = -\frac{d\epsilon}{dp} = \frac{\alpha}{p+\beta},$$

and

$$k = ac = \frac{c\alpha}{p+\beta},$$

$$\frac{1}{a}\frac{dk}{dz} = \frac{c}{a}\frac{da}{dz} = -\frac{c}{p+\beta}\frac{dp}{dz},$$

$$t' = \frac{\gamma}{q}(h-z) \qquad \text{and} \qquad w = \gamma(h-z) - p,$$

Therefore

$$\frac{dp}{dt'} = -\frac{q}{\gamma}\frac{dp}{dz} \qquad \text{and} \qquad \frac{dw}{dz} = -\gamma - \frac{dp}{dz}, \qquad \frac{d^2w}{dz^2} = -\frac{d^2p}{dz^2}.$$

Substituting in (16) we get

$$-\frac{q}{\gamma}\frac{dp}{dz} = c\left[\frac{d^2p}{dz^2} - \frac{1}{p+\beta}\frac{dp}{dz}\left(\gamma + \frac{dp}{dz}\right)\right].$$

or

$$c\frac{d^2p}{dz^2} + \left(\frac{q}{\gamma} - \frac{c\gamma}{p+\beta}\right)\frac{dp}{dz} - \frac{c}{p+\beta}\left(\frac{dp}{dz}\right)^2 = 0. \qquad (42)$$

This differential equation is of a type seldom encountered, and therefore we will give all the steps required for its solution.

First, let $m = \frac{dp}{dz}$, then $\frac{d^2p}{dz^2} = m\frac{dm}{dp}$ and we get

$$cm\frac{dm}{dp} + \left(\frac{q}{\gamma} - \frac{c\gamma}{p+\beta}\right)m - \frac{c}{p+\beta}m^2 = 0.$$

or

$$\frac{dm}{dp} + \left(\frac{q}{\gamma c} - \frac{\gamma}{p+\beta}\right) - \frac{m}{p+\beta} = 0.$$

Let now $m = m_0 + u$ where m_0 is a constant at our disposal

to which we will give such a value as to simplify our differential equation. Substituting, we have:

$$\frac{du}{dp} - \frac{1}{p+\beta}u = \frac{\gamma}{p+\beta} - \frac{q}{\gamma c} + \frac{m_0}{p+\beta}.$$

Take

$$m_0 = -\gamma,$$

Then

$$\frac{du}{dp} = \frac{1}{p+\beta}u = -\frac{q}{\gamma c}.$$

Let now $u = xy$ where both x and y are variables. Therefore

$$\left(\frac{dy}{dp} - \frac{1}{p+\beta}y\right) + \frac{1}{x}\left(y\frac{dx}{dp} + \frac{q}{\gamma c}\right) = 0.$$

Now let y be such that

$$\frac{dy}{dp} - \frac{1}{p+\beta}y = 0.$$

A solution is $y = p + \beta$. With this value of y the differential equation becomes

$$(p+\beta)\frac{dx}{dp} = -\frac{q}{\gamma c},$$

which has

$$x = -\frac{q}{\gamma c}\log R(p+\beta),$$

as a solution, R being a constant of integration.

$$u = xy = -\frac{q}{\gamma c}(p+\beta)\log R\,(p+\beta),$$

and

$$\frac{dp}{dz} = m = m_0 + u = -\gamma - \frac{q}{\gamma c}(p+\beta)\log R(p+\beta),$$

or

$$\frac{dp}{dz} + \frac{q}{\gamma c}(p+\beta)\log R(p+\beta) + \gamma = 0. \qquad (43)$$

The boundary conditions are, as before,

$$z = h : p = w = 0,$$

$$z = 0 : \frac{dw}{dz} = 0 \quad \text{or} \quad \frac{dp}{dz} = -\gamma.$$

If we call p_1 the value of p at the bottom of the deposit ($z = 0$) where $dp/dz = -\gamma$ we will have from (43) the following condition:

$$\frac{q}{\gamma c}(p_1 + \beta) \log R(p_1 + \beta) = 0.$$

Since β is essentially positive and R must be finite, we must have

$$R(p_1 + \beta) = 1, \tag{44}$$

Let $r = \log R(p + \beta)$. Then the solution of (43) is

$$\frac{1}{A}\int \frac{dr}{\frac{r}{A} + e^{-r}} = -\frac{q}{\gamma c}(z + S), \tag{45}$$

where S is a constant of integration and

$$A = \frac{R\gamma^2 c}{q}.$$

In order to evaluate the integral in (45), expand e^{-r} in a power series, and then the integral will take the form

$$\frac{1}{A}\int \frac{dr}{\frac{r}{A} + 1 - r + \frac{r^2}{2!} - \frac{r^3}{3!} + \cdots}.$$

If r is small we can disregard the terms in the series development containing r in powers higher than the second. With this assumption only an approximate solution can be obtained but such a solution is not expected to differ appreciably from the exact

solution, since for small values of z (where the variation of p is greater) r is very small, it being zero when $z = 0$ on account of (44) if (44) is still to hold true after the above assumption is made. (This will be shown later to be the case).

Equation (45) then becomes:

$$\frac{1}{A}\int \frac{dr}{1-\left(1-\frac{1}{A}\right)r+\frac{r^2}{2}} = -\frac{q}{c}(z+S),$$

or

$$\frac{1}{A}\left[\frac{1}{\sqrt{\left(1-\frac{1}{A}\right)^2-2}}\log\frac{r-\left(1-\frac{1}{A}\right)-\sqrt{\left(1-\frac{1}{A}\right)^2-2}}{r-\left(1-\frac{1}{A}\right)+\sqrt{\left(1-\frac{1}{A}\right)^2-2}}\right] = -\frac{q}{\gamma c}(z+S), \quad \text{(A)}$$

if $\left(1-\frac{1}{A}\right)^2 > 2$, and

$$\frac{1}{A}\left[\frac{2}{\sqrt{2-\left(1-\frac{1}{A}\right)^2}}\tan^{-1}\frac{r+\left(1-\frac{1}{A}\right)}{\sqrt{2-\left(1-\frac{1}{A}\right)^2}}\right] = -\frac{q}{\gamma c}(z+S) \quad \text{(B)}$$

if $\left(1-\frac{1}{A}\right)^2 < 2$.

We will assume solution (A) to hold, and justify this later.

Substituting in (A) the values of r and A and introducing the boundary conditions, $dp/dz = -\gamma$ for $z = 0$ where $p = p_1$, and $p = 0$ for $z = h$, we get:

$$[\log R(p_1+\beta)]^2 - 2\left(1-\frac{q}{R\gamma^2 C}\right)\log R(p_1+\beta) - \frac{2}{R(p_1+\beta)} = -2.$$

One solution of this equation is $R(p_1 + B) = 1$ which is in accordance with equation (44).

The equation of the *pressure* distribution is

$$\frac{\log R(p+\beta) - \left(1 - \frac{q}{R\gamma^2 c}\right) - \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}}{\log R(p+\beta) - \left(1 - \frac{q}{R\gamma^2 c}\right) + \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}} = \left[e^{-R\gamma\sqrt{[1-(q/R\gamma^2 c)]^2-2}s}\right]\left[e^{-R\gamma\sqrt{[1-(q/R\gamma^2 c)]^2-2}S}\right]. \quad (46)$$

The constant p_1 is determined from (44) and R and S from the following equations:

$$\frac{\log R\beta - \left(1 - \frac{q}{R\gamma^2 c}\right) - \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}}{\log R\beta - \left(1 - \frac{q}{R\gamma^2 c}\right) + \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}} = \frac{\left(1 - \frac{q}{R\gamma^2 c}\right) + \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}}{\left(1 - \frac{q}{R\gamma^2 c}\right) - \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}} e^{-R\gamma\sqrt{[1-(q/R\gamma^2 c)]^2-2}s}, \quad (47)$$

$$e^{-R\gamma\sqrt{[1-(q/R\gamma^2 c)]^2-2}S} = \frac{2}{\left[-\left(1 - \frac{q}{R\gamma^2 c}\right) + \sqrt{\left(1 - \frac{q}{R\gamma^2 c}\right)^2 - 2}\right]^2}. \quad (48)$$

By working out the solution in terms of trigonometric functions, (solution (B)), and determining the constants R, S and p_1, it was found that the condition expressed by equation (44) does not hold, and therefore it does not represent the solution of our problem.

Equation (47) can be solved only by trial. In order to make the solution of equation (47) easier we will work out a less accurate solution by setting $e^{-r} = 1 - r$ which is not much in error for small values of r. The equation thus obtained will be

less complex, and will show approximately the values of R to be tried in equation (47). We have

$$\frac{1}{A}\int \frac{dr}{\frac{r}{A}+1-r} = -\frac{q}{\gamma c}(z+S). \tag{49}$$

Integrating and simplifying we get

$$\left(1-\frac{q}{R\gamma^2 c}\right)\log R(p+\beta) = 1 - e^{-[(q/\gamma c)-R\gamma](z+S)}.$$

Introducing the boundary conditions, we have

$$\left(1-\frac{q}{R\gamma^2 c}\right)\log R(p_1+\beta) + \frac{1}{R(p_1+\beta)} = 1,$$

one solution of which is $R(p_1+\beta)=1$, and this is also in accordance with (44).

Also $S=0$,

$$\left(1-\frac{q}{R\gamma^2 c}\right)\log R\beta = 1 - e^{-[(q/\gamma c)-R\gamma]h}, \tag{50}$$

and

$$\left(1-\frac{q}{R\gamma^2 c}\right)\log R(p+\beta) = 1 - e^{-[(q/\gamma c)-R\gamma]z}. \tag{51}$$

R can be found from (50) and then values of p for several values of z are given by (51).

(b) *Second Stage.* In this stage, since t is independent of z, the differential equation is partial instead of ordinary and if it can be solved at all the resulting equation will be so complex as to make the analysis worthless. As a matter of fact, a comparison between the results obtained by applying equations (27) and (46) to a concrete problem, which will be given later, conclusively shows that they differ but slightly, and therefore equation (27) is accurate enough for any practical purpose. The same would, of course, be true for the second stage of consolidation. Consequently, there is no need of trying to solve equation (16) for the second stage.

So far we have determined the pressure distribution with reference to the reduced depth of the deposit. We will now show how it can be determined with reference to the actual or true thickness of the deposit, and also its settlement at any time.

4. Actual Pressure Distribution and Settlements. Let H be the actual thickness of the deposit which corresponds to the reduced thickness h. Let also Z be the distance of a point in the actual deposit measured from the bottom surface, and corresponding to the reduced distance z. Then

$$H = \int_0^h (1 + \epsilon)dz, \tag{52}$$

and

$$Z = \int_0^z (1 + \epsilon)dz. \tag{53}$$

where, as before, ϵ is the voids-ratio and measures the water content per unit of solid matter.

In general we know the true depth of the deposit and what we want to find is its reduced thickness. If ϵ is known for several depths, then a curve can be plotted and the integrals (52) and (53) calculated graphically.

Now it was pointed out that there is a definite relation between the water content and the granular pressure for any given material and this is given by

$$\epsilon = -\alpha_1 \log (p + \beta) - \alpha_2(p + \beta) + C_1,$$

or, since α_2' is very small, we may write

$$\epsilon = -\alpha \log (p + \beta) + C_2.$$

Now p can be expressed as a function of z: For the first stage of consolidation it is expressed by equation (27) and for the second stage by equation (40) where $p = \gamma(h - z) - w$, the variable t being kept constant during the integration since this is performed with respect to z only.

Therefore

$$Z = \int_0^z [-\alpha \log (p + \beta) + C]dz, \tag{54}$$

where $C = 1 + C_2$.

Having thus obtained Z as a function of z, and the pressure distribution in terms of z, the latter will be determined in terms of Z by simply changing the ordinates by the transformation (54). In practice, however, the variation of ϵ with z is small and practically uniform so that an average value of ϵ may be taken and introduced in equation (53). Let ϵ_0 be the average value of ϵ, then

$$Z = (1 + \epsilon_0)z, \tag{55}$$

which is a linear transformation and is equivalent to

$$Z = \frac{H}{h}z. \tag{55a}$$

To compute the settlement of the top surface of the deposit, let H_0 and ϵ_0 respectively, be the true depth and the voids-ratio at time $t = 0$, i.e., just after sedimentation has stopped. Let H_1 and ϵ_1 be the corresponding quantities at any subsequent time $t = t_1$. If we denote by s the settlement of the top surface at any time t, we will have $s_1 = H_0 - H_1$ as the total settlement of the top surface of the deposit in the time interval t_1.

Now the time rate of change in water content per unit of solid matter is $\partial\epsilon/\partial t$ which is a negative quantity since ϵ decreases with the time and therefore the time rate of change in water content in a layer of thickness dz located at distance z above the bottom surface is

$$-\frac{\partial \epsilon}{\partial t}dz = a\frac{\partial p}{\partial t}dz.$$

Hence the rate at which the top surface of the deposit is settling at any specified time $t = t_1$ is

$$\left(\frac{ds}{dt}\right)_{t=t_1} = \int_0^z a\left(\frac{\partial p}{\partial t}\right)_{t=t_1} dz. \tag{56}$$

Therefore the total settlement which takes place in a time interval from $t = 0$ to $t = t_1$ is

$$s_1 = \int_0^{t_1} \int_0^h a \left(\frac{\partial p}{\partial t} \right) dz\, dt = \int_0^{t_1} \left(\frac{ds}{dt} \right) dt, \tag{57}$$

and that from a time $t = t_1$ to a time $t = t_2$ is

$$S^1 = \int_{t_1}^{t_2} \int_0^h a \left(\frac{\partial p}{\partial t} \right) dz\, dt.$$

Introducing the value of $\frac{\partial p}{\partial t} \left(= -\frac{\partial w}{\partial t} \right)$ from equation (40) in equations (56) and (57) and keeping a constant during the integrations, we get:

$$\frac{ds}{dt} = 8aqh \sum_{n=1,\,3,\,\cdots} \frac{\frac{2hb}{n\pi} \sin \frac{n\pi}{2} + e^{-bA}}{4h^2b^2 + n^2\pi^2} e^{-(cn^2\pi^2/4h^2)t}, \tag{56a}$$

and

$$s = \int_0^t \frac{ds}{dt}\, dt$$

$$= \sum_{n=1,\,3,\,5\,\cdots} \frac{32a^2qh^3}{k\pi^2} \frac{\frac{2hb}{n\pi} \sin \frac{n\pi}{2} + e^{-bA}}{n^2(4h^2b^2 + n^2\pi^2)} (1 - e^{-(cn^2\pi^2/4h^2)t}), \tag{57a}$$

as the rate of settlement and total settlement of the top of the deposit at any time, t.

At this point it is well to point out the fact that the coefficient of permeability K, as used in the previous equations is in terms of the reduced dimensions, while in performing the experiment already described, its value was obtained in terms of the true dimensions of the layer of material under test. In applying the law of Darcy, let $Q' = k'i'A'$ refer to the true dimensions of the layer and $Q = kiA$ to the reduced dimensions. Then, since $Q = Q'$ and $A = A'$ we must have

$$k'i' = ki,$$

or

$$k = k' \frac{\bar{v}'}{i} = k' \frac{\frac{h}{l}}{\frac{h(l+\epsilon)}{l}} = \frac{k'}{l+\epsilon}, \tag{58}$$

according to Fig. 1 where l is the true thickness of the layer and h the hydraulic head.

If the average value of ϵ is again introduced, we shall have

$$k = \frac{h}{H} k'. \tag{58a}$$

We are now in a position to apply our formulae to a specific problem, but before we do so we will investigate the meaning of some of the previous equations.

Equation (27) shows that the smaller b ($b = qa/\gamma k$) is, the greater will be p for a given value of z. Hence, of two mud deposits for which a, γ and k are the same, the one for which q is smaller (q measures the rate at which solid matter is being sedimented) will be in a more consolidated state than the other. This fact deserves some consideration for frequently, at the same basin, for some reason or other, sedimentation is much more intensive in some particular locations than in others, and the fact that the material deposited is the same throughout the basin would lead to the erroneous conclusion that the state of internal stresses should be the same throughout the basin. Also, other things being equal, the greater is the coefficient of permeability, the more consolidated will be the deposit.

Equation (26) shows that the state of internal stress is independent of the hydrostatic head under which the deposit is being formed, and merely depends on the thickness of the deposit. This is evident because the excess in hydrostatic pressure, i.e., the hydrodynamic pressure is what produces the flow of water within the material.

Equation (40) shows that the time rate of change of the hydrodynamic stress, $(\partial w/\partial t)$, decreases with the time in the

same rate as the hydrodynamic stress itself. The slope $(\partial w/\partial z)$ of the $w(z)$ curve for a given value of z is continuously decreasing as time goes on, and therefore the quantity of water percolating through a given section of the deposit per unit time decreases with the time. This means that the variation of p with the time is greater at the beginning of the process of consolidation than for large values of t, i.e., the consolidation is more effective at the beginning of the (second) stage.

5. Example. We shall now take up the problem of determining the state of internal stress and settlements of a delta deposit advancing at a given constant speed towards the ocean as is given on p. 177 of Terzaghi's "*Erdbaumechanik.*" The consolidation of the deposit is due only to its own weight. The deposit is advancing towards the ocean at a rate of 1 m. per year and its true depth is 50 m.; the plane on which sedimentation takes place has a slope of 1 to 10 and therefore the time required for the formation of a layer of 50 m. is 500 years. The specific weight of the material sedimented is 2.7 grams per cubic centimeter, and this gives $\gamma = 2.7 - 1 = 1.7$ grams per cubic centimeter. The average value of the voids-ratio is 1.0 and therefore the reduced thickness of the deposit is 25 m. The rate of sedimentation is then $10/2 = 5$ cubic centimeter per year and per unit of area. Therefore:

$$q = 5 \times 1.7 = 8.5 \text{ gr/cm}^2 \text{ per year.}$$

The true average value of the coefficient of permeability is 1.63 cm./year and therefore its reduced value is

$$k = \frac{1.63}{1 + \epsilon_a} = \frac{1.63}{2} = 0.815 \text{ cm./year.}$$

The average value of the modulus of compression is

$$a = 0.00024 \text{ cm.}^5/\text{gr.}$$

Having this information we are in a position to determine the pressure distribution throughout the deposit.

(a) *First Stage.* Let $h' = 10$ m. $= 1{,}000$ cm.

$$b = \frac{qa}{\gamma k} = \frac{8.5 \times 0.00024}{1.7 \times 0.815} = 0.001472,$$

$$c = \frac{k}{a} = 3{,}395,$$

$$\frac{\gamma}{b} = \frac{1.7}{0.001472} = 1{,}153,$$

$$e^{-bh'} = 0.2305,$$

$$\gamma h = 1.7 \times 2{,}500 = 4{,}250 \text{ gr./cm.}^2.$$

From equation (26) we obtain the following values of p and w for $z = 0$, and $z = 500$ cms., respectively:

$$p = 1{,}153\ (1 - 0.2305) = 887 \text{ gr./cm.}^2$$

$$w = 813 \text{ gr./cm.}^2$$

$$p = 1{,}153\ (0.4790 - 0.2305) = 287 \text{ gr./cm.}^2$$

$$w = 563.$$

At the end of the first stage of consolidation, ($h = 2{,}500$ cms.) we have the following pressure distribution:

z cms.	0	500	1,000	1,500	2,000	2,500
p gr./cm.^{2}	1,125	525	237	110	30	0
w gr./cm.^{2}	3,125	2,875	2,313	1,590	820	0

(b) *Second Stage.* In this case we will compute the pressures for $t = 500$, 1,000 and 5,000 years, the time being measured from the beginning of the stage, i.e., after the sediment has reached a height equal to h.

Equation (40) may be written in the form:

$$w = \frac{16\gamma h^2 b}{\pi} \sum_{n=1,\,3,\,5,\,\cdots} N_n e^{-(cn^2\pi^2/4h^2)} \sin \frac{n\pi}{2h}(h - z),$$

where

$$N_n = \frac{\frac{2hb}{n\pi} \sin \frac{n\pi}{2} + e^{-bh}}{n(4h^2b^2 + n^2\pi^2)},$$

$$\frac{16\gamma h^2 b}{\pi} = 79{,}700.$$

The results may now be tabulated as follows:

n	N_n	$e^{-(cn^2\pi^2/4h^2)t}$			$\sin\frac{n\pi}{2h}(h-z)$			
		$t=500$	$t=1{,}000$	$t=5{,}000$	$z=0$	$z=\frac{1}{4}h$	$z=\frac{1}{2}h$	$z=\frac{3}{4}h$
1	+0.0370	0.512	0.262	0.00122	+1	+0.924	+0.707	+0.383
3	−0.00264	0.00238	0.0000061	—	−1	−0.216	+0.707	+0.924
5	+0.000820	—	—	—	+1	−0.383	−0.707	+0.924

From this table we can compute the values of

$$N_n e^{-(cn^2\pi^2/4h^2)t} \sin\frac{n\pi}{2h}(h-z)$$

which are given in the following table.

Values of z	$t=500$		$t=1{,}000$		$t=5{,}000$	
	$n=1$	$n=3$	$n=1$	$n=3$	$n=1$	$n=3$
0	+0.0190	$+6.28\times10^{-6}$	+0.00969	$+1.61\times10^{-6}$	+0.0000451	—
$\frac{1}{4}h$	+0.0175	$+1.36\times10^{-6}$	+0.00895	$+0.348\times10^{-6}$	+0.0000416	—
$\frac{1}{2}h$	+0.0136	-4.44×10^{-6}	+0.00685	-1.14×10^{-6}	+0.0000319	—
$\frac{3}{4}h$	+0.00726	-5.80×10^{-6}	+0.00371	-1.49×10^{-6}	+0.0000173	—

All the computations were carried through in order to show how rapidly the series involved in equation (40) converges. From the above table we see that the error introduced by neglecting the terms for which $n = 3, 5 \ldots$ is, in all cases, much less than one in one thousand. Therefore equation (41) may be used to advantage in all cases, except for very small values of t.

The following table gives the values of p and w for the above values of t and z

Values of z	Values of w in gr./cm.^{3}			Values of p in gr./cm.^{3}		
	$t=500$	$t=1,000$	$t=5,000$	$t=500$	$t=1,000$	$t=5,000$
0	1,510	771	3.6	2,740	3,479	4,246
$\frac{1}{4}h$	1,395	713	3.3	1,803	2,483	3,194
$\frac{1}{2}h$	1,083	546	2.5	1,042	1,579	2,123
$\frac{3}{4}h$	579	296	1.4	484	767	1,061

The times 500, 1,000 and 5,000 years correspond respectively to points in the deposit at the distances 500, 1,000 and 5,000 m. from the coast. From the above table we see that at a point 5 km. from the coast the deposit is practically consolidated. It should be remembered that so far we have studied only the process of consolidation of the deposit under the action of its own weight, and evaporation together with other irregular phenomena occurring at the top surface of the deposit play an important part during the process of consolidation as will be shown later.

No data are available to compute the water content throughout the deposit, but, at least for the first stage of consolidation, we can take

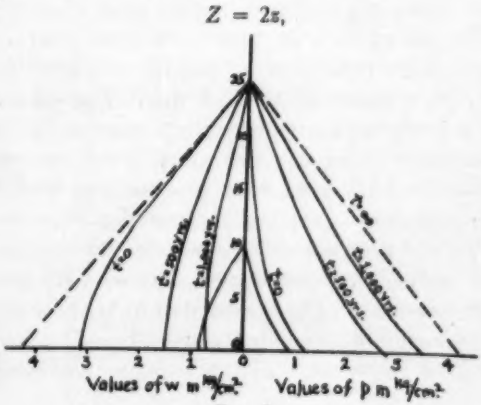

FIG. 9.

and by this single transformation obtain the actual pressure curves.

The results given in the above table are shown graphically in Fig. 9.

(c) *Settlements and Rates of Settlements.* These are found from equations (56a) and (57a) which were obtained for the case of a = constant.

If, however, there is an appreciable variation in the values of a, we have, as already pointed out,

$$a = \frac{\alpha}{p + \beta},$$

where α and β are constants. Then

$$\frac{ds}{dt} = \int_0^h \frac{\alpha}{p + \beta} \frac{\partial p}{\partial t} dz,$$

and

$$s = \int_0^t \frac{ds}{dt} dt.$$

The above integrations would lead to such complex expressions as to be of no practical value and therefore it seems that if the variation of a is to be accounted for, it is better to compute a for several values of t and then find the rates of settlement for these values of t and a from (56a). Then the total settlement during any given time interval can be computed by simple additions. It is however, believed that equations (56a) and (57a) are accurate enough for any practical purpose.

Equation (56a) shows that the rate at which the top surface of the deposit is settling decreases exponentially with the time, it being a maximum at the beginning of the stage ($t = 0$) and zero for $t = \infty$. Equation (57a) shows that the total settlement of the top surface increases exponentially with the time and is zero at the beginning of the stage ($t = 0$). All of these facts have been actually observed in Nature.

We will now apply (56a) and (57a) to the previous example

for t = 0, 50, 100, 500, 1,000, 2,000, 3,000 and 5,000 years. The results are given in the following table, and represented graphically in Fig. 10.

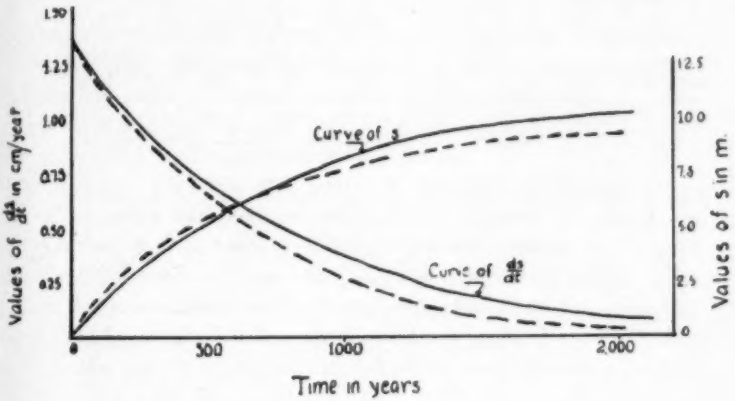

FIG. 10.

Values of t years	0	50	100	500	1,000	2,000	3,000	5,000	∞
$\frac{ds}{dt}$ in cm./year	1.35	1.26	1.20	0.71	0.40	0.105	0.0276	0.00184	0
s in m.....	0	0.611	1.29	5.27	8.30	10.25	10.82	11.00	11.05

Both equations (56a) and (57a) depend directly on a since $k = ca$ or, in other words, the constants b and c are not affected by the variation in a. Now a decreases with the time and therefore the curve of ds/dt should be lower for large values of t, but this does not indicate that the curve of s would be lowered in the same proportion, for it depends also on the previous values of a. Thus, if for t = 2,000 years the value of a were one-half of that used in computing the above table, we would have curves as those shown dotted in Fig. 10.

We can compute Z in terms of z and find the actual pressure

distribution for given values of t. Since we do not know the values of α and β for the material of the deposit in question, we will use equation (55a) instead of equation (54). The results are as follows:

Values of t years	0	50	100	500	1,000	5,000
Ratio Z/z or H/h	2	1.976	1.948	1.779	1.668	1.560

(d) *Solution by Applying Equations* (46) *and* (51). In order to obtain the pressure distribution we need the value of β. This is not known. We will therefore proceed to find the variation of the pressures p with β. The computations are very tiresome, and will not be given here. The results obtained by applying equation (51) are given in the following table, where p_1 is the granular pressure at the bottom surface of the deposit.

R	β	p_1	Maximum Error in Per Cent
0.00010	8,820	1,180	0.72
20	3,770	1,230	3.31
25	2,750	1,250	—
30	2,060	1,270	8.46
36	1,480	1,295	—
40	1,200	1,300	16.8
50	685	1,315	18.7

The last column indicates the maximum error introdueed by placing $e^{-r} = 1 - r$. This of course does not mean that the results obtained deviate from the exact ones by the same amounts, since r is not a constant but a variable, no fixed relation existing between the two errors in question.

The above results are illustrative in that they show how little p_1 depends on β. β was made to vary from 8,820 to 685 while the corresponding values of p were found to be 1,180 and 1,315.

We turn now to the more accurate solution represented by

equation (46). As already pointed out, this equation represents the solution of our problem only when

$$\left(1 - \frac{q}{Ry^2c}\right)^2 > 2.$$

This shows that R in this case must be less than about 0.000359. For this value of R the equation does not hold, the pressure curve being a triangle. p_1 was computed from equation (46) for two values of R and the results are as follows:

R	β	p_1
0.00010	8,820	1,180
0.00030	1,960	1,370

The maximum errors introduced by placing

$$e^{-r} = 1 - r + \frac{r^2}{2}$$

are in both cases very small.

The pressures for other values of z were computed the results being shown in Fig. 11.

Now the values of β are in general, much smaller than those for which p_1 was computed, so that it seems that there would be wider discrepancies than those shown by the preceding curves. But, on the other hand, in order to compare the results of the two theories, α and β must be such as to make the average value of a, as computed from

$$a = \frac{\alpha}{p + \beta}$$

compatible with the average value of 0.00024. The value of a (0.00024) was computed from a range in voids-ratio of from 1.2 to 0.8[9] which correspond respectively to the voids-ratio at the top surface where $p = 0$ and the bottom surface where $p = p_1$.

[9] Terzaghi's "Erdbaumechanik," p. 178, see also p. 141.

Hence

$$\frac{1.2 - 0.8}{p_1 - 0} = 0.00024 \quad \text{or} \quad p_1 = 1{,}668 \text{ gr./cm.}^2$$

$$1.2 = -\alpha \log (0 + \beta) + C.$$

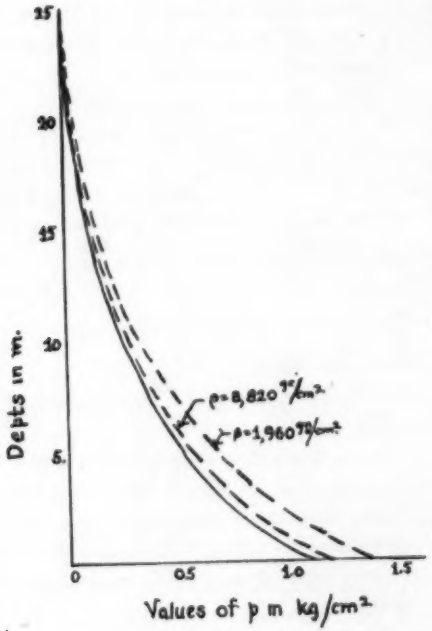

FIG. 11.

Also

$$0.8 = -\alpha \log (p_1 + \beta) + C.$$

Therefore

$$\alpha = \frac{0.4}{\log\left(1 + \frac{p_1}{\beta}\right)} = \frac{0.4}{\log\left(1 + \frac{1{,}668}{\beta}\right)}.$$

If then, a varies according to the above law, α and β must be such that the average value of a, as p is made to vary from 0 to 1,668, will be

$$\frac{1}{2}\left(\frac{\alpha}{\beta}+\frac{\alpha}{1{,}668+\beta}\right)=0.00024.$$

Combining the two last equations, we get:

$$\frac{0.4}{\log\left(1+\frac{1{,}668}{\beta}\right)}(1{,}668+2\beta)-0.8-0.00048\beta^2=0.$$

Solving this equation, we get:

$$\beta = 3{,}730 \qquad \text{and} \qquad \alpha = 1.06.$$

If the pressure distribution is now computed from equation (46) for the above value of β the resulting $p(z)$ curve will be closer to that represented by equation (27) than the one shown in the preceding figure for which $\beta = 1{,}960$ gr./cm.^{2}. This leads to the conclusion that the theory developed for a constant is far more accurate than one could ever expect. Hence, the differential equation (16) can be entirely dropped out of consideration, and equation (15) used instead. This will be done in what follows.

IV. Problem 2. Mud Deposit under the Influence of a Very Permeable Fill, Placed on Top of it at the Beginning of the Second Stage of Consolidation.

1. Determination of Stresses. The consolidation due to the weight of the material itself has already been considered in the preceding section. The combined effect of the weight of the material itself and that of the top fill will be ascertained by properly combining the solution obtained for the two cases separately. Therefore in order to study the behavior of the mud deposit under the influence of the top fill alone, we disregard the weight of the material and proceed as follows, after neglecting the time required for depositing the fill, and also the resistance of the fill against percolation.

Let h be the reduced thickness of the deposit, and let p and w, be the granular and the hydrodynamic pressures respectively at any section distant z (or x) from the bottom (or top) surface at any particular time t.

Let $t = 0$ correspond to the instant at which sedimentation has just ceased and let the fill be deposited at this same instant.

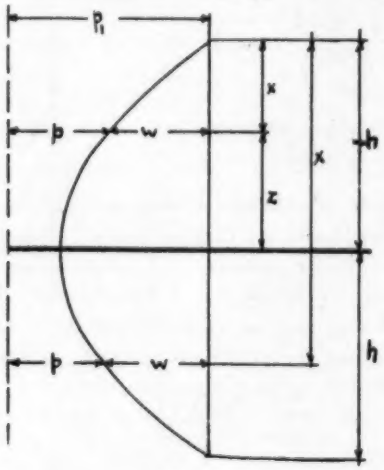

FIG. 12.

Let p_1 be the pressure per unit area exerted by the fill resting on the top of the deposit.

Since the resistance of the fill against percolation of water is neglected, the hydrodynamic pressure at the surface of the deposit will be zero and the granular pressure will be equal to p_1 (Fig. 12). In order to have equilibrium we must have the following relation between p and w.[10]

$$p + w = p_1. \tag{59}$$

As before, we have

$$\frac{\partial p}{\partial t} = - c \frac{\partial^2 w}{\partial z^2},$$

[10] Cf. Terzaghi, "Erdbaumechanik," pp. 143–144.

and from (59) we get

$$\frac{\partial w}{\partial t} = c\frac{\partial^2 w}{\partial z^2} = c\frac{\partial^2 w}{\partial x^2}, \tag{60}$$

where $x = h - z$.

No water can possibly flow during the period of time in which the fill is deposited, (which is practically equal to zero and so considered) and therefore we must have $w = p_1$ throughout the deposit for $t = 0$.

Hence the boundary conditions are

$$\begin{aligned} w &= 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad t > 0, \\ \frac{\partial w}{\partial x} &= 0 \quad \text{for} \quad x = h, \\ w &= p_1 \quad \text{for} \quad t = 0. \end{aligned} \tag{61}$$

The second of the above boundary conditions is equivalent to

$$w = 0 \quad \text{for} \quad x = 2h,$$

as already pointed out.

Let

$$w = e^{\alpha t + \beta z}$$

be the solution of (60). Then

$$\alpha = c\beta^2 = -c\lambda^2.$$

therefore

$$w = e^{-c\lambda^2 t}\cos \lambda x, \tag{62}$$

$$w = e^{-c\lambda^2 t}\sin \lambda x, \tag{63}$$

are particular solutions of (60).

Now (63) is seen to satisfy both $w = 0$ for $x = 0$, and $w = 0$ for $x = 2h$, provided we set

$$\lambda = \frac{n\pi}{2h},$$

where $n = 1, 2, 3, \ldots$.

Hence the general solution of the differential equation (60) is given by

$$w = \sum_{n=1}^{n=\infty} a_n e^{-(cn^2\pi^2/4h^2)t} \sin \frac{n\pi}{2h} x, \tag{64}$$

where the a_n represent constants multiplying every term of the series.

But

$$w = p_1 \qquad \text{for} \qquad t = 0,$$

and therefore

$$p_1 = \sum_{n=1}^{n=\infty} a_n \sin \frac{n\pi}{2h} x,$$

or

$$a_n = \frac{2}{2h} \int_0^{2h} p_1 \sin \frac{n\pi}{2h} x dx,$$

or

$$a_n = \frac{2p_1}{n\pi} (1 - \cos n\pi),$$

and this is zero for even values of n and equal to $4p_1/n\pi$ for odd values of n.

Hence we have

$$w = \frac{4}{\pi} p_1 \sum_{n=1, 3, \cdots} \frac{e^{-(cn^2\pi^2/4h^2)t} \cos \frac{n\pi}{2h} z}{n \sin \frac{n\pi}{2}}, \tag{65}$$

and

$$p = p_1 \left(1 - \frac{4}{\pi} \sum_{n=1, 3, \cdots} \frac{e^{-(cn^2\pi^2/4h^2)t} \cos \frac{n\pi}{2h} z}{n \sin \frac{n\pi}{2}} \right). \tag{66}$$

With regard to the boundary conditions we made use of two conditions which, at first, seem to be inconsistent, namely: $w = 0$ for $x = 0$ and $w = p_1$ for $t = 0$ throughout the deposit. Now, at the very surface of the deposit the granular pressure must always be equal to p_1, while at any other section, p_1 is taken up partly by the capillary water and partly by the granular

material (according to (59)), but at $t = 0$, when p_1 is supposed to be applied, no water can possibly flow on account of the smallness of the value of the coefficient of permeability, and since water is practically incompressible, it follows that at $t = 0$ the whole pressure p_1 must be taken up by the capillary water. Consequently there is a discontinuity in the pressure distribution at the surface of the deposit at the instant $t = 0$.

The assumption that $w = p_1$ for $t = 0$ may be readily confirmed by applying Heaviside's Operational Method [11] to the differential equation (60). Since this method affords a simple way of solving the above differential equation and at the same time no assumption concerning w is involved, it will be given below.

We will now use the differential equation in terms of p instead of w. It is

$$\frac{\partial p}{\partial t} = c \frac{\partial^2 p}{\partial z^2}. \tag{67}$$

The boundary conditions are

$$p = p_1 \quad \text{for} \quad z = h, \tag{68}$$

and

$$\frac{\partial p}{\partial z} = 0 \quad \text{for} \quad z = 0. \tag{69}$$

We now introduce an operator in (67), i.e., we set

$$\frac{\partial}{\partial t} = r.$$

Then

$$c \frac{d^2 p}{dz^2} = rp,$$

or

$$\frac{d^2 p}{dz^2} = \frac{r}{c} p = -\lambda^2 p, \tag{70}$$

[11] Cf. "Heaviside's Operators in Engineering and Physics," by E. J. Berg, *Journal of the Franklin Institute*, Nov. 1924. Also: "An Analogy between Pure Mathematics and Operational Mathematics of Heaviside by Means of the Theory of H-Functions," by J. J. Smith, *Journal of the Franklin Institute*, Oct., Nov., and Dec., 1925.

where $\lambda^2 = -r/c$, λ being a function of r (i.e. of t) but not of z.

The solution of (70) is

$$p = C_1 \cos \lambda z + C_2 \sin \lambda z,$$

where C_1 and C_2 are functions of r.

Introducing the boundary conditions (68) and (69) we have

$$C_2 = 0,$$

and

$$C_1 = \frac{p_1}{\cos \lambda h}.$$

Hence

$$p = p_1 \frac{\cos \lambda z}{\cos \lambda h} 1, \tag{71}$$

where 1 is the so-called "unit function."

Equation (71) may be written as follows

$$p = p_1 \frac{Y(r)}{Z(r)} 1, \tag{72}$$

where $Y(r)$ and $Z(r)$ are functions of r.

The Solution of equation (72) is given by Heaviside in the form of a series which is called the "Expansion Theorem." It is as follows

$$p = p_1 \left[\frac{Y_{(0)}}{Z_{(0)}} + \sum_{r=r_1, r_2, \cdots} \frac{Y_{(r)} e^{rt}}{r \dfrac{dZ_{(r)}}{dr}} \right], \tag{73}$$

r_1, r_2, $\cdots$, etc. being the roots of the equation $Z_{(r)} = 0$, and $Y_{(0)}$ and $Z_{(0)}$ being the values of $Y_{(r)}$ and $Z_{(r)}$ when r is zero.

We can now apply the expansion theorem (73) to equation (71).

We have

$$Y_{(r)} = \cos \lambda z \qquad \text{and} \qquad Z_{(r)} = \cos \lambda h.$$

The roots of $Z_{(r)} = 0 : \cos \lambda h = 0$ are

$$\lambda = \frac{n\pi}{2h} \qquad (n = 1, 3, 5, \cdots).$$

But

$$r = -c\lambda^2.$$

Therefore

$$r = -\frac{cn^2\pi^2}{4h^2}. \tag{74}$$

Now $\lambda = 0$ when $r = 0$ and therefore

$$\frac{Y_{(0)}}{Z_{(0)}} = \left(\frac{\cos \lambda x}{\cos \lambda h}\right)_{\lambda=0} = 1. \tag{75}$$

$$r\frac{dZ_{(r)}}{dr} = c\lambda^2 \frac{dZ_{(r)}}{d\lambda}\frac{d\lambda}{dr} = -\frac{n\pi}{4}\sin\frac{n\pi}{2}. \tag{76}$$

Substituting (74), (75) and (76) together with the value of $Y(r)$ in the expansion theorem (73) we have:

$$p = p_1\left[1 - \frac{4}{\pi}\sum_{n=1,3,\cdots}^{\infty} \frac{e^{-(cn^2\pi^2/4h^2)t}\cos\frac{n\pi}{2h}z}{n\sin\frac{n\pi}{2}}\right],$$

which is the same as equation (66).

Before we take up any specific problem we will investigate how the state of internal stresses is affected by the presence of evaporation at the surface of the deposit.

2. Consolidation by Evaporation. Evaporation of water at the surface of the deposit produces surface tension at the entrances of the voids, which in turn brings the water within the voids into a state of tensile stress. The intensity of the stress depends on the temperature and degree of humidity of the atmosphere and also upon the velocity of the wind at the surface. We will not describe here the phenomenon of evaporation,[13] but just

[13] Cf. Terzaghi, "Erdbaumechanik," pp. 137–139, 162.

point out that it produces tension in the capillary water, which in turn affects the state of stresses in the granular material.

Let w_1 be the intensity of tension existing in the capillary water at the surface of the deposit. Since there is equilibrium the hydrodynamic and the granular pressures must be equal and opposite in sign, in every section of the deposit. It follows then that at the top surface there must be granular pressure equal to w_1 but opposite in sign. Hence, although no granular pressure exists at the very surface of the deposit, there will always exist a pressure p_1 ($= -w_1$) at an infinitesimal distance below the surface. We have here the same type of discontinuity in the pressure distribution at the surface as in the case of the permeable top fill.

The problem of determining the effect of evaporation on the process of consolidation of a mud deposit is largely indeterminate for the following reasons: first w_1 is variable and does not seem to follow any definite law, its value varying from zero up to values higher than 100 kg./cm.^{2}; second, as evaporation becomes intensive, the water withdraws from the surface towards the interior of the deposit (as is the case when the quantity of water percolating upwards from the interior of the deposit is less than that which is being evaporated), thus forming a more compact layer of material at the surface which is less permeable than the remainder of the deposit.

If, however, we assume a constant value for w_1, which may be taken as the average value during a certain period of time the solution will be represented by equation (66) or

$$w = -p = -w_1 \left[1 - \frac{4}{\pi} \sum_{n=1,\,3,\,5} \frac{e^{-(cn^2\pi^2/4h^2)t} \cos \frac{n\pi}{2h} z}{n \sin \frac{n\pi}{2}} \right]. \quad (77)$$

3. Settlements. The time rate of settlement of the surface of the deposit due to a granular pressure p_1 at the surface can be

found from equation (56). It is

$$\frac{ds}{dt} = \frac{2kp_1}{h} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-(cn^2\pi^2/4h^2)t}. \tag{78}$$

This converges very rapidly and therefore we can take

$$\frac{ds}{dt} = \frac{2kp_1}{h} e^{-(c\pi^2/4h^2)t}, \tag{78a}$$

unless t is small, in which case (78) should be used.

The total settlement at any particular time t, is:

$$s = \frac{8ap_1h}{\pi^2} \sum_{n=1,3,5} \frac{(1 - e^{-(cn^2\pi^2/4h^2)t})}{n^2}. \tag{79}$$

If t is large we can use the following formula which gives sufficient accuracy

$$s = \frac{8ap_1h}{\pi^2}(1.052 - e^{-(c\pi^2/4h^2)t}). \tag{79a}$$

In both cases a was considered constant during the integrations.

4. Example. We will now investigate the conditions of stress and settlements of the mud deposit of Problem I, under the influence of a permeable fill, exerting a pressure of 1 kg./cm.^{2} on the top surface of the deposit.

We have

$$c = 3{,}395, \qquad p_1 = 1{,}000,$$
$$a = 0.00024, \qquad h = 2{,}500.$$

The results of the computations are shown in Figures 13 to 17. Fig. 13 shows the stress distribution for the case of a top permeable fill exerting a pressure of 1,000 gr/cm^2 at the top surface of the deposit.

Fig. 14 shows the stress distribution due to the above pressure, and that of the weight of the solid matter itself.

Fig. 15 shows the influence of evaporation for a value of w_1 equal to 10 kg/cm^2. The value of w_1 is in general much larger

and variable, but this serves the purpose of showing the general shape of the curves.

Fig. 16 shows the rates of settlement and total settlements due to the weight of the top fill.

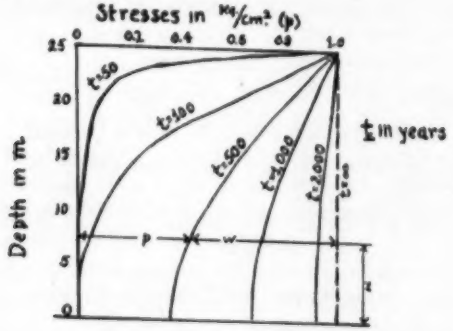

FIG. 13.

Fig. 17 shows these same two items combined with those due to the weight of the deposit.

The shape of the curves in Fig. 14 is seen to agree with that of experimental curves obtained by Dr. C. Terzaghi and published

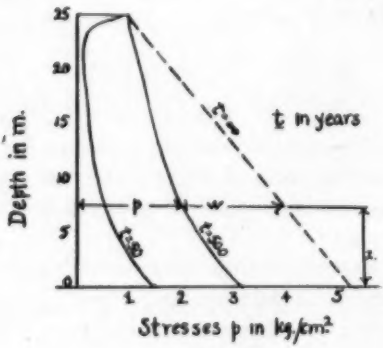

FIG. 14.

in the Journal of the Boston Society of Civil Engineers (Vol. XII, No. 10, Dec. 1925).

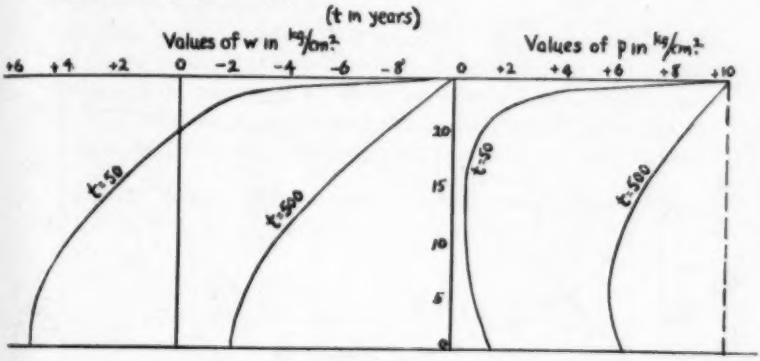

FIG. 15.

V. *Problem* 3. MUD DEPOSIT HAVING WITHIN IT A THIN LAYER OF LESS PERMEABLE MATERIAL

We will assume here that at a height l from the bottom of the deposit whose depth is h, there exists a layer of material of thickness $(m - l)$ having a coefficient of permeability smaller than that of the remainder of the deposit. (See Fig. 18.)

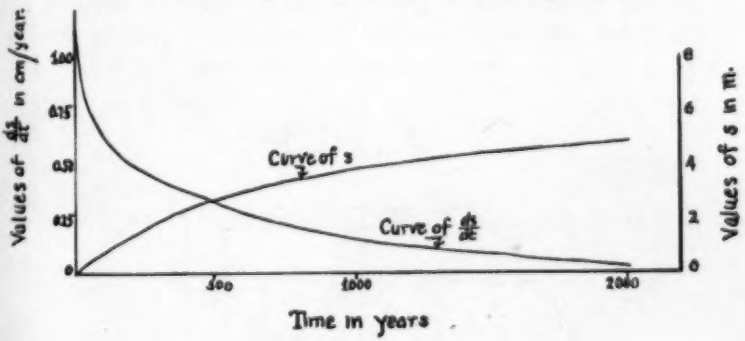

FIG. 16.

In a problem of this sort, we can not expect to have an accurate solution by applying the simple differential equation

$$\frac{\partial p}{\partial t} = - c \frac{\partial^2 w}{\partial z^2},$$

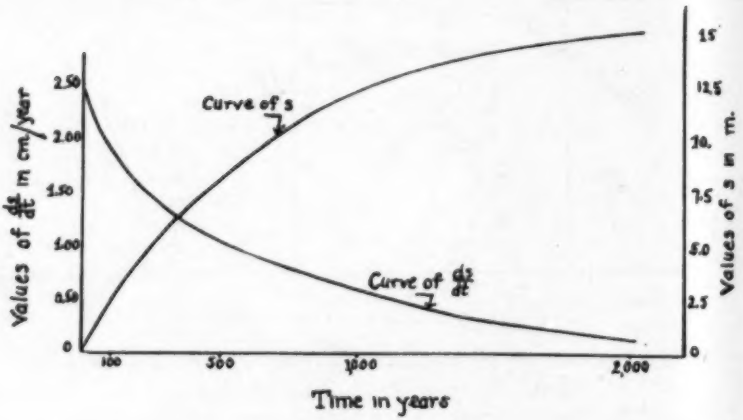

FIG. 17.

since the conditions are too variable throughout the deposit. If we attempt to use the more complex differential equation (16) the problem will be beyond solution. Even by applying the above differential equation [equation (15)], the mathe-

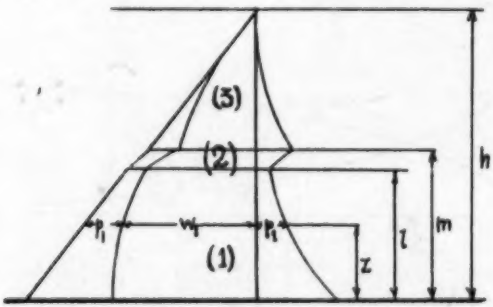

FIG. 18.

matical analysis becomes so complicated that we are forced to make some approximations.

In Fig. 18 the deposit is shown divided into three layers. Let the characteristic constants (already defined) for layers (1) and (3) be γ, q, a, and k and let those for layer (2) be γ_1, q_1, a_1, and k_1. Let also the granular and the hydrodynamic pressures in in these layers be respectively p_1, p_2, p_3 and w_1, w_2, and w_3.

1. **First Stage.** In this stage the differential equation can be solved without any difficulty.

We have

$$\frac{\partial p_1}{\partial t'} = c\frac{\partial^2 w_1}{\partial z^2} \quad \text{for} \quad 0 \leq z \leq l, \tag{80}$$

$$\frac{\partial p_2}{\partial t'} = c_1\frac{\partial^2 w_2}{\partial z^2} \quad \text{for} \quad l \leq z \leq m, \tag{81}$$

$$\frac{\partial p_3}{\partial t'} = c\frac{\partial^2 w_3}{\partial z^2} \quad \text{for} \quad m \leq z \leq h. \tag{82}$$

The values of t' as functions of z corresponding to the above differential equations are respectively,

$$t' = \frac{\gamma}{q}(l - z) + \frac{\gamma_1}{q_1}(m - l) + \frac{\gamma}{q}(h - m), \qquad 0 \leq z \leq l,$$

$$t' = \frac{\gamma_1}{q_1}(m - z) + \frac{\gamma}{q}(h - m), \qquad l \leq z \leq m,$$

and

$$t' = \frac{\gamma}{q}(h - z), \qquad m \leq z \leq h.$$

Hence the differential equations (80), (81), and (82) become:

$$\frac{d^2 p_1}{dz^2} + b\frac{dp_1}{dz} = 0, \tag{80a}$$

$$\frac{d^2 p_2}{dz^2} + b_1\frac{dp_2}{dz} = 0, \tag{81a}$$

$$\frac{d^2p_2}{dz^2} + b\frac{dp_2}{dz} = 0, \tag{82a}$$

where

$$b = \frac{q}{\gamma c},$$

and

$$b_1 = \frac{q_1}{\gamma_1 c_1}.$$

The solutions of (80a), (81a) and (82a) are, respectively,

$$p_1 = A + Be^{-bz}, \tag{83}$$

$$p_2 = C + De^{-b_1 z}, \tag{84}$$

and

$$p_3 = E + Fe^{-b_2 z}, \tag{85}$$

where A, B, C, D, E and F are constants to be determined.

The relations between the w's and the p's are:

$$w_1 = \gamma(h - m) + \gamma_1(m - l) + \gamma(l - z) - p_1, \tag{83a}$$

$$w_2 = \gamma(h - m) + \gamma_1(m - z) - p_2, \tag{84a}$$

$$w_3 = \gamma(h - z) - p_3, \tag{85a}$$

therefore

$$\frac{dw_1}{dz} = -\gamma - \frac{dp_1}{dz},$$

$$\frac{dw_2}{dz} = -\gamma_1 - \frac{dp_2}{dz},$$

and

$$\frac{dw_3}{dz} = -\gamma - \frac{dp_3}{dz}.$$

Remembering that the quantity of water (per unit of time) which leaves layer (1) is the same as that which enters layer (2) (the same thing being true for layers (2) and (3)), we will have the following boundary conditions:

For $z = 0 : \frac{dw_1}{dz} = 0,$ (a)

For $z = h : p_3 = 0,$ (b)

For $z = l : p_1 = p_2,$ (c)

For $z = m : p_2 = p_3,$ (d)

For $z = l : k\frac{dw_1}{dz} = k_1\frac{dw_2}{dz},$ (e)

For $z = m : k_1\frac{dw_2}{dz} = k\frac{dw_3}{dz}.$ (f)

Introducing these boundary conditions we get the following relations between the constants A, B, C, D, E and F:

$$-Bb = -\gamma, \quad (a')$$

$$E + Fe^{-bh} = 0, \quad (b')$$

$$A + Be^{-bl} = C + De^{-b_1 l}, \quad (c')$$

$$E + Fe^{-bm} = C + De^{-b_1 m}, \quad (d')$$

$$-k\gamma + kbBe^{-bl} = -k_1\gamma_1 + k_1 b_1 De^{-b_1 l}, \quad (e')$$

$$-k_1\gamma_1 + k_1 b_1 De^{-b_1 m} = -k\gamma + kbFe^{-bm}. \quad (f')$$

Solving these six equations as simultaneous, we get:

$$B = \frac{\gamma}{b},$$

$$D = \frac{k_1\gamma_1 + k\gamma(e^{-bl} - 1)}{k_1 b_1 e^{-b_1 l}},$$

$$F = \frac{1}{kbe^{-bm}}\left[k\gamma e^{-bl} - \frac{k_1\gamma_1 - k\gamma(1 - e^{-bl})}{e^{-b_1 l}}\left(e^{-b_1 l} - e^{-b_1 m}\right)\right],$$

$$E = -Fe^{-bh},$$

$$A = \left(\frac{k\gamma}{k_1 b_1} - \frac{\gamma}{b}\right)e^{-bl} - F\left[e^{-bh} - e^{-bm}\left(1 - \frac{kb}{k_1 b_1}\right)\right],$$

$$C = E + Fe^{-bm} - De^{-b_1 m}.$$

Hence the distribution of pressures throughout the deposit is fully determined for the first stage.

2. Second Stage. In this stage, since t is independent of z, the three differential equations will be partial instead of ordinary. Unfortunately these differential equations can not be solved and therefore we will take

$$\frac{\partial p}{\partial t} = -r\frac{\partial^2 w}{\partial z^2},$$

or

$$\frac{\partial w}{\partial t} = r\frac{\partial^2 w}{\partial z^2}, \tag{86}$$

as the general equation for the second stage, where r stands for c for layers (1) and (3), and for c_1 for layer (2). r is then variable but will be considered constant in order to make the problem solvable.

Changing the origin of coordinates to the top surface (86) becomes

$$\frac{\partial w}{\partial t} = r\frac{\partial^2 w}{\partial x^2}, \tag{86a}$$

where $x = h - z$.

We now proceed as in the case of the mud deposit of homogeneous material.

The boundary conditions are

$$w = 0 \quad \text{for} \quad x = 0,$$

and

$$w = 0 \quad \text{for} \quad x = 2h,$$

the last condition resulting from the fact that

$$\frac{\partial w}{\partial x} = 0.$$

The conditions are now represented by Fig. 19, where the pressure distribution is symmetric with respect to the impermeable surface.

The solution of equation (86a) is given, as before, by:

$$w = \sum_{n=1}^{n=\infty} A_n e^{-(n^2\pi^2/4h^2)t} \sin \frac{n\pi}{2h} x, \tag{87}$$

or

$$w = \sum_{n=1}^{n=\infty} A_n e^{-(n^2\pi^2/4h^2)t} \frac{\cos \frac{n\pi}{2h} z}{\sin \frac{n\pi}{2}}. \tag{87a}$$

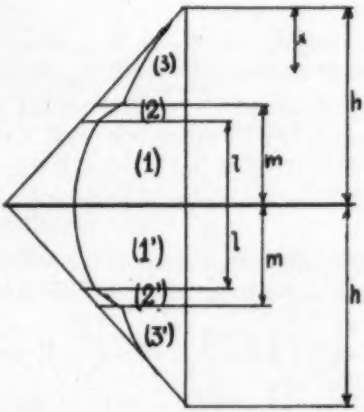

FIG. 19.

When $t = 0$, $w = \varphi(x)$, where $\varphi(x)$ indicates the pressure distribution at the end of the first stage and is, therefore, a discontinuous function, but always finite. Therefore its integration should be performed by parts.

Since for $t = 0$, $w = \varphi(x)$, we must have

$$A_n = \frac{1}{h} \int_0^{2h} \varphi(x) \sin \frac{n\pi}{2h} x dx.$$

The expressions for $\varphi(x)$ are as follows (see Fig. 19):

For layer (3) : $w_3 = \gamma x - E - Fe^{-b(h-x)}$.

For layer (2) : $w_2 = (\gamma - \gamma_1)(h - m) + \gamma_1 x - C - De^{-b_1(h-x)}$.

For layer (1) : $w_1 = (\gamma_1 - \gamma)(m - l) + \gamma x - A - Be^{-b(h-x)}$.

For layer (1′) : $w'_1 = (\gamma_1 - \gamma)(m - l) - \gamma x - A - Be^{-b(x-h)}$.

For layer (2′) : $w'_2 = \gamma(h - m) + \gamma_1(h + m) - \gamma_1 x - C - De^{-b_1(x-h)}$.

For layer (3′) : $w'_3 = 2\gamma h - \gamma x - E - Fe^{-b(x-h)}$.

Therefore

$$A_n h = \int_0^{h-m} w_3 \sin\frac{n\pi}{2h}\,xdx + \int_{h-m}^{h-l} w_2 \sin\frac{n\pi}{2h}\,xdx + \int_{h-l}^{h} w_1 \sin\frac{n\pi}{2h}\,xdx + \int_{h}^{h+l} w_1' \sin\frac{n\pi}{2h}\,xdx + \int_{h+l}^{h+m} w_2' \sin\frac{n\pi}{2h}\,xdx + \int_{h+m}^{2h} w_3' \sin\frac{n\pi}{2h}\,xdx,$$

where the values of the w's are given by the preceding equations.

Performing the above integrations and simplifying we get

$$A_n = \frac{4h}{n^2\pi^2}(\gamma - \gamma_1)\left[\left(2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right)\left(\cos\frac{n\pi}{2h}m - \cos\frac{n\pi}{2h}l\right) + \frac{n\pi}{h}\sin\frac{n\pi}{2}\left(m\sin\frac{n\pi}{2h}m - l\sin\frac{n\pi}{2h}l\right)\right] + \frac{8\gamma h}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\left\{2\sin\frac{n\pi}{2}\left[(E - C)\sin\frac{n\pi}{2h}\cdot m + (C - A)\sin\frac{n\pi}{2h}l\right] + E(\cos n\pi - 1) - (\gamma - \gamma_1) \times \left[2\sin\frac{n\pi}{2}\left(m\sin\frac{n\pi}{2h}m - l\sin\frac{n\pi}{2h}l\right) + 2h\cos\frac{n\pi}{2} \times \left(\cos\frac{n\pi}{2h}l - \cos\frac{n\pi}{2h}m\right)\right]\right\} - \frac{4h}{4h^2b^2 + n^2\pi^2} \times \left\{2\sin\frac{n\pi}{2}\left[Fe^{-bm}\left(b\cos\frac{n\pi}{2h}m - \frac{n\pi}{2h}\sin\frac{n\pi}{2h}m\right)\right.\right.$$

$$- Be^{-bl}\left(b\cos\frac{n\pi}{2h}l - \frac{n\pi}{2h}\sin\frac{n\pi}{2h}l\right)\Big]$$
$$+ Fe^{-bh}\frac{n\pi}{2h}(1 - \cos n\pi) + 2Bb\sin\frac{n\pi}{2}\Bigg\}$$
$$- \frac{4h}{4h^2b_1^2+n^2\pi^2}\Bigg\{2D\sin\frac{n\pi}{2}\Big[e^{-b_1l}\left(b_1\cos\frac{n\pi}{2h}l - \frac{n\pi}{2h}\sin\frac{n\pi}{2h}l\right)$$
$$- e^{-b_1m}\left(b_1\cos\frac{n\pi}{2h}m - \frac{n\pi}{2h}\sin\frac{n\pi}{2h}m\right)\Big]\Bigg\}. \tag{88}$$

Now once the values of the trigonometric functions are tabulated, expression (88) is not very hard to compute, but it seems to be too long to be of any practical value.

In case the thickness of layer (2) is not large, we can set $l = m$ and get a much simpler equation for A_n. This will be shown later, by an example, to involve a very small error. Also A_n is seen to be small for *even* values of n, the prevailing term in (88) being $8\gamma h/n^2\pi^2 \sin n\pi/2$.

Setting $l = m = l'$ (where l' is now the average between l and m) in (88) we get:

$$A_n = \frac{8\gamma h}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\Bigg[2\sin\frac{n\pi}{2}(E - A)\sin\frac{n\pi}{2h}l'$$
$$+ E(\cos n\pi - 1)\Bigg] - \frac{4h}{4h^2b^2 + n^2\pi^2}\Bigg[2(F - B)\sin\frac{n\pi}{2}e^{-bl'}$$
$$\times\left(b\cos\frac{n\pi}{2h}l' - \frac{n\pi}{2h}\sin\frac{n\pi}{2h}l'\right)$$
$$+ F\frac{n\pi}{2h}e^{-bh}(1 - \cos n\pi) + 2Bb\sin\frac{n\pi}{2}\Bigg]. \tag{89}$$

A still more roughly approximate expression is

$$A_n = \frac{8\gamma h}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{8hBb}{4h^2b^2 + n^2\pi^2}\sin\frac{n\pi}{2}. \tag{90}$$

It should be remembered that the assumption or approximation $l = m$ was made only in order to obtain a simpler expression for A_n and should not affect the constants A, B, C, etc.

The fact that the state of stress for the first stage of consolidation can be determined furnishes valuable information. For the second stage of consolidation, we can obtain only an approximate solution, and therefore we will not enter into many details with regard to the mathematical analysis.

We will now determine the value of r which will represent, approximately, the state of stress in layer (1). To do this, we know that the quantity of water which leaves layer (1) per unit time must be the same as that which enters layer (2) per unit of time. If a solution could be obtained for this stage, we would have three stress equations—one for each layer, and instead of r, we would have c for layers (1) and (3), and c_1 for layer (2). Assuming that equation (87) represents the solution of our problem is equivalent to assuming that

$$k\left(\frac{\partial w}{\partial z}\right)_{z=l,\ r=c} = K_1\left(\frac{\partial w}{\partial z}\right)_{z=l,\ r=c_1}.$$

Although this is not true, it affords a means of estimating the value of r with the following modifications: Compute values of $k(\partial w/\partial z)_{z=l,\ r=c}$ for several values of t, within a certain period of time, and then compute the corresponding values of $k_1(\partial w/\partial z)_{z=l,\ r=c_1}$ for the same values of t. Now take the average of each pair of values for the same value of t which then represents the quantity of water percolating through the section $z = l$ per unit of time and per unit of area. After this is done we determine the value of r, say r_1, which will make the same average values between

$$k\left(\frac{\partial w}{\partial z}\right)_{z=l,\ r=r_1} \quad \text{and} \quad k_1\left(\frac{\partial w}{\partial z}\right)_{z=l,\ r=r_1}$$

approximately the same as the average values already obtained.

Another method which may be used to determine r is the following. The rate at which the granular stress p is increasing with the time $(\partial p/\partial t')$ at the end of the first stage of consolidation for sections far away from the top surface of the deposit, should

be about the same as its rate of increase at the beginning of the second stage. In other words,

$$-c\left(\frac{d^2w}{dz^2}\right)_{z=z_1} = -r\left(\frac{\partial^2 w}{\partial z^2}\right)_{z=z_1,\ t=0}.$$

The right hand member of this equation can be computed only by taking $t > 0$, and finding its limit as t is made to approach zero, otherwise a very large number of terms would be required in the series (87). The method of procedure is: compute d^2w/dz^2 for, say, $z = 0$ and $z = l$ (equation (83)). Then compute $\partial^2 w/\partial z^2$ (equation (87)) for the same values of z and for several values of t, say, 50,100, and 500 years by assigning to r a trial value. With these values, plot curves between $(\partial^2 w/\partial z^2)_{z=z_1}$ and t and produce them to meet the axis $t = 0$. r should then be such as to satisfy the above equation.

In all cases r must be $c_1 < r < c$.

We are not interested in the stress distribution in layer (2) which may be considered as a plane of separation between layers (1) and (3).

The stress distribution in layer (3) can not be determined unless we make some approximations. In this case, however, we know the limiting values between which the pressures must lie. Let the granular and hydrodynamic stress for this layer be p_3 and w_3, respectively. The two limiting values for w_3 are given by equation (87) for $r = r_1$ and $r = c$ for $0 \leq x \leq h - m$. They are

$$w_3' = \sum_{n=1}^{n=\infty} A_n e^{-(n^2\pi^2 r_1/4h^2)t} \sin\frac{n\pi}{2h}x, \tag{91}$$

and

$$w_3'' = \sum_{n=1}^{n=\infty} A_n e^{-(n^2\pi^2 c/4h^2)t} \sin\frac{n\pi}{2h}x. \tag{91a}$$

Now the change in water-content of layer (2) is very small, and can be neglected. Therefore we can assume that the quantity of water which leaves layer (1) per unit time is the

same as that which enters layer (3) per unit of time. This leads to the equation:

$$\left(\frac{\partial w_1}{\partial z}\right)_{z=l} = -\left(\frac{\partial w_3}{\partial x}\right)_{x=h-m}, \tag{92}$$

where $\partial w_3/\partial x$ has an intermediate value between $\partial w'/\partial x$ and $\partial w''/\partial x$. This equation serves either as a check on the results or as a means of determining to which of the equations (91) and (91a) the pressure w_3 is closer.

3. Settlements and Rates of Settlements. These can be determined by directly applying equations (56) and (57). Let w_3 be the average between w'_3 and w''_3 and let l' be the distance of the center of layer (2) from the bottom of the deposit. Then, taking a constant, we have

$$\frac{ds}{dt} = a\int_0^{l'} \frac{\partial p_1}{\partial t}\,dz + a\int_0^{l'} \frac{\partial p_3}{\partial t}\,dx$$

$$= -a\int_0^{l'} \frac{\partial w_1}{\partial t}\,dz - \frac{a}{2}\int_0^{l'} \left(\frac{\partial w_3'}{\partial t} + \frac{\partial w_3''}{\partial t}\right) dx.$$

Integrating, we will have for the rate of settlement,

$$\frac{ds}{dt} = \frac{a\pi}{4h}\sum_{n=1}^{n=\infty} \left\{ re^{-(n^2\pi^2/4h^2)t}\left(1 + 2\sin\frac{n\pi}{2}\sin\frac{n\pi}{2h}l' - \cos\frac{n\pi}{2h}l'\right)\right.$$

$$\left. + ce^{-(cn^2\pi^2/4h^2)t}\left(1 - \cos\frac{n\pi}{2h}l'\right)\right\} A_n n. \tag{93}$$

The total settlement of the top surface of the deposit at any time t is:

$$s = \frac{ah}{\pi}\sum_{n=1}^{n=\infty} \frac{A_n}{n}\left\{\left(1 + 2\sin\frac{n\pi}{2}\sin\frac{n\pi}{2h}l' - \cos\frac{n\pi}{2h}l'\right)\right.$$

$$\left.\times\left(1 - e^{-(n^2\pi^2/4h^2)t}\right) + \left(1 - \cos\frac{n\pi}{2h}l'\right)\left(1 - e^{-(cn^2\pi^2/4h^2)t}\right)\right\}. \tag{94}$$

In computing the values of ds/dt and s from (93) and (94) for very small values of t, two or more terms should be considered.

For large values of t, only one term in the above series will give results accurate enough for any practical purpose.

4. Example. We will now investigate the state of stresses in a mud deposit with a total reduced thickness of 25 m. and having within it a layer of less permeable material with a reduced thickness of 50 cm. Such a large thickness for this layer is chosen in order to give an idea of the discrepancies in the values of A_n as computed from equations (89) and (90).

The characteristic constants of the materials are[13]:

Layers (1) and (3): $\gamma = 1.7, \quad a = 0.00024, \quad k = 0.815.$

Layer (2): $\gamma_1 = 1.9, \quad a_1 = 0.00050, \quad k_1 = 0.20.$

Take $q = q_1 = 8.5$. Therefore

$$c = \frac{k}{a} = 3{,}395, \qquad b = \frac{q}{\gamma c} = 0.001472,$$

and

$$c_1 = \frac{k_1}{a_1} = 400, \qquad b = \frac{q_1}{\gamma_1 c_1} = 0.01118.$$

The reduced dimensions are

$$l = 1{,}200 \text{ cm.}, \quad m = 1{,}250 \text{ cm.} \quad \text{and} \quad h = 2{,}500 \text{ cm.}$$

It would take too much space if we were to reproduce here all the computations involved in the determination of the state of stress in each layer. Therefore we will illustrate only the steps in which some doubt might arise in the interpretation of the equations.

(*a*) *First Stage.* There is nothing in particular to be said about the determination of p and w for this stage. We first compute the values of the constants A, B, C, D, E and F. Then the granular pressures are determined from equations (83), (84) and (85) and the hydrodynamic pressures from equations (83a),

[13] Cf. Terzaghi, "Erdbaumechanik," Fig. 30, p. 171. Also "Principles of Soil Mechanics," *Engineering News Record*, Nov. 26, 1925.

(84a) and (85a). The results of the computations are graphically represented by Fig. 20.

(*b*) *Second Stage.* The values of A_n, computed from equations (88), (89) and (90) for $n = 1$, $n = 2$ and $n = 3$ are

Formula (88): $A_1 = 2{,}820$, $A_2 =$ practically zero.

Formula (89): $A_1 = 2{,}825$, $A_2 = 0$, $A_3 = -330$.

Formula (90): $A_1 = 2{,}930$, $A_2 = 0$, $A_3 = -150$.

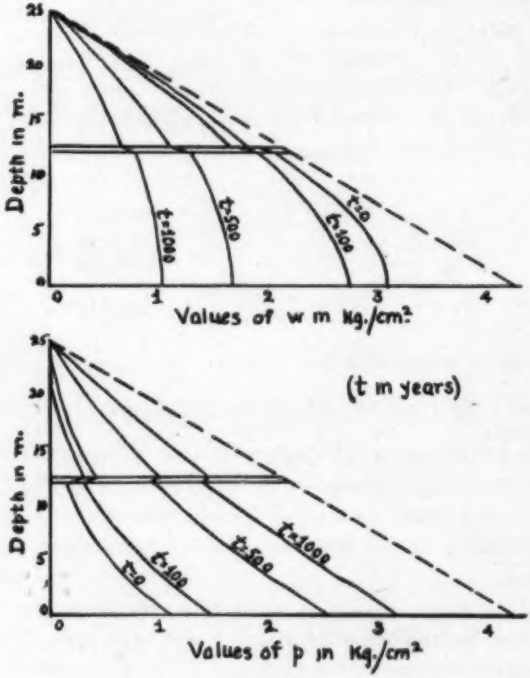

FIG. 20.

This shows that, for $n = 1$, either formula (89) or (90) may be used to compute A_n, while for large values of n, formula (89) should be used.

Determination of r: The average values between $k(\partial w/\partial z)_{z=l,\ r=c}$ and $k_1(\partial w/\partial z)_{z=l_1,\ r=c_1}$, for $t_1 = 100$, 500 and 1,000 years are, respectively,

0.91, 0.68 and 0.46 cm.^{3}/year.

The value of r, (r_1), which gives average values of $k(\partial w/\partial z)_{z=l_1,\ r=r_1}$ and $k_1(\partial w/\partial z)_{z=l_1,\ r=r_1}$ closer to the above one is $r_1 = 2{,}600$. These average values for the above values of are, respectively,

0.91 +, 0.71 and 0.42 cm.^{3}/year.

The second method gives a value for r_1 of about 2000 +.

We will therefore take $r_1 = 2{,}300$. This value is not too large because part of the influence exerted by the more impermeable layer is already included in the values of A_n.

The results of computations are shown graphically in Fig. 20 for $t = 0$, 100, 500 and 1,000 years.

VI. Problem 4. Consolidation of Mud Deposits by Drainge

1. General. We have seen how slow is the process of consolidation of mud deposits (or of fine-grained materials) when under the influence of the weight of the material itself. This is due to the extremely low value of the coefficient of permeability of fine-grained soils. It is evident that in order to effectively drain a mud deposit, we must have at least one layer of coarser material (like sand) within the deposit. If, in the mud deposit discussed in Problem III, the interposed layer had been composed of a coarser material like sand, the hydrodynamic pressure, at the bottom and top of the layer, would be the same because of the extremely high value of the coefficient of permeability of sand compared with that of mud or clay. Hence, when a layer of very permeable material (which we may call sand) comes between the layers of mud, this layer can be entirely disregarded in the computation of the stresses, using as reduced dimension, that of the deposit without the sand layer.

2. Determination of Stresses. Consider now a mud deposit of total reduced thickness h, interposed by a sand layer whose center lies at a distance (reduced) l from the bottom surface which we assume horizontal and impermeable.

Let us now suppose that a pipe is driven into the deposit to reach the layer of sand and that the water is discharged at the top surface of the deposit. Under these conditions, the pressure in the water at the section $z = l$ is hydrostatic, i.e., the hydrodynamic pressure is zero. Let $\tau = 0$ correspond to the instant at which drainage begins. This instant is a short time after the pipe has been driven, to account for the rapid change in the hydrodynamic pressure in the sand layer itself.

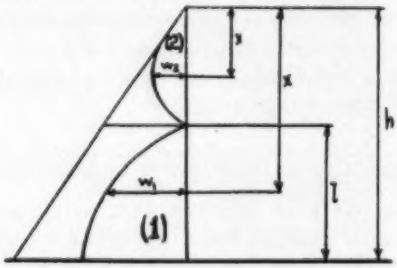

FIG. 21.

The boundary conditions are (Fig. 21):

$$\frac{\partial w}{\partial z} = 0 \qquad \text{for} \qquad z = 0,$$

$$w = 0 \qquad \text{for} \qquad z = l,$$

$$w = 0 \qquad \text{for} \qquad z = h,$$

and

$$w = f(z) \qquad \text{for} \qquad \tau = 0.$$

We will need to consider only the case in which the deposit is in its second stage of consolidation. The effect of drainage at a particular location of the deposit is effective over a considerable distance from it on account of the high value of the coefficient

of permeability of the sand, unless the sand layer is discontinuous.

For $\tau = 0$, and for any value of t, the pressure distribution is given by equation (38) or

$$w = \sum_{n=1,3,\cdots} a_n T \sin \frac{n\pi}{2h} x, \tag{95}$$

where

$$T = e^{-(cn^2\pi^2/4h^2)t}, \tag{96}$$

and a_n is given by equation (39).

τ is not a function of T, because t refers to the time before drainage has started.

Let p_1, w_1 and p_2 be the granular and hydrodynamic pressures at the portion of the deposit below and above the sand layer, respectively (see Fig. 21). Let also

$$a_n T = a_n e^{-(cn^2\pi^2/4h^2)t} = K_n, \tag{97}$$

then, for $\tau = 0$,

$$w = \sum_{n=1,3,\cdots} K_n \sin \frac{n\pi}{2h} x. \tag{95a}$$

We have now to solve two differential equations, namely:

$$\frac{\partial w_1}{\partial t} = c \frac{\partial^2 w_1}{\partial x^2}, \tag{98}$$

and

$$\frac{\partial w_2}{\partial \tau} = c \frac{\partial^2 w_2}{\partial x^2}. \tag{99}$$

(*a*) *Solution of equation* (99). This equation is subjected to the following boundary conditions:

$$w_2 = 0 \quad \text{for} \quad x = 0 \tag{100}$$

$$w_2 = 0 \quad \text{for} \quad x = h - l, \tag{101}$$

and

$$w_2 = w = f(x), \quad \text{for} \quad \tau = 0$$

and

$$0 \leq x \leq h - l. \tag{102}$$

The particular solutions of (99) are, as before,

$$w_2 = e^{-c\lambda^2\tau} \sin \lambda x, \tag{103}$$

and

$$w_2 = e^{-c\lambda^2\tau} \cos \lambda x. \tag{104}$$

Now the particular solution (103) will satisfy both (100) and (101), for all values of τ, provided that we take

$$\lambda = \frac{m\pi}{h - l} \qquad m = 1, 2, 3, \ldots .$$

Hence the general solution of (99) is

$$w_2 = \sum_{m=1}^{m=\infty} R_m e^{-[cm^2\pi^2/(h-l)^2]} \sin \frac{m\pi}{h - l} x. \tag{105}$$

To determine R_m we have that for $\tau = 0$,

$$w_2 = \sum_{n=1, 3, \ldots} K_n \sin \frac{n\pi}{2h} x = f(x).$$

or

$$\sum_{m=1}^{m=\infty} R_m \sin \frac{m\pi}{h - l} x = \sum_{n=1, 3, \ldots} K_n \sin \frac{n\pi}{2h} x. \tag{106}$$

Therefore

$$R_m = \frac{2}{h - l} \int_0^{h-l} \sum_{n=1, 3, \ldots} K_n \sin \frac{n\pi}{2h} x \sin \frac{m\pi}{h - l} x dx. \tag{107}$$

Integrating, we get

$$R_m = \frac{2\pi}{(h - l)^2} \sum_{n=1, 3, \ldots} K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi \cos \frac{n\pi}{2h} l}{\left(\frac{n\pi}{2h}\right)^2 - \left(\frac{m\pi}{h - l}\right)^2}. \tag{108}$$

Therefore

$$w_2 = \frac{2\pi}{(h - l)^2} \sum_{m=1}^{m=\infty} \sum_{n=1, 3, \ldots} \left[K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi \cos \frac{n\pi}{2h} l}{\left(\frac{n\pi}{2h}\right)^2 - \left(\frac{m\pi}{h - l}\right)^2} \right] \times \left[e^{-[cm^2\pi^2/(h-l)^2]\tau} \sin \frac{m\pi}{h - l} x \right], \tag{109}$$

and

$$p_2 = \gamma x - w_2. \tag{109a}$$

(*b*) *Solution of equation* (98). The solution has to hold for $h - l \leq x \leq h$. Let (Fig. 22)

$$y = x - h + l \quad \text{or} \quad x = y + h - l,$$

then

$$\frac{\partial w_1}{\partial \tau} = c \frac{\partial^2 w}{\partial y^2}. \tag{98a}$$

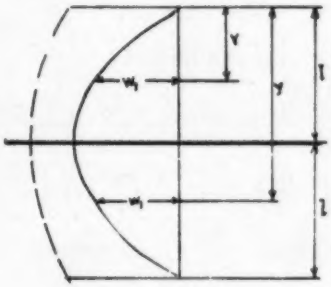

FIG. 22.

The boundary conditions are:

$$w_1 = 0 \quad \text{for} \quad y = 0 \tag{110}$$

$$\frac{\partial w_1}{\partial y} = 0 \quad \text{for} \quad y = l \quad \text{or} \quad w_1 = 0 \quad \text{for} \quad y = 2l, \tag{111}$$

$$w_1 = \varphi(y), \quad \text{for} \quad \tau = 0 \quad \text{and} \quad 0 \leq y \leq l.$$

This latter condition reduces to

$$\varphi_1(y) = \sum_{n=1,3,\cdots} K_n \sin \frac{n\pi}{2h} (y + h - l) \quad \text{for} \quad 0 \leq y \leq l,$$

and

$$\varphi_2(y) = \sum_{n=1,3,\cdots} K_n \sin \frac{n\pi}{2h} (h + l - y) \quad \text{for} \quad l \leq y \leq 2l.$$

Now the particular solution (103):

$$w_1 = e^{-\alpha\lambda^2} \sin \lambda y,$$

will satisfy both (110) and (111), if we set

$$\lambda = \frac{m\pi}{2l}, \qquad m = 1, 2, 3, \cdots.$$

Hence the general solution of (98a) is:

$$w_1 = \sum_{m=1}^{m=\infty} S_m e^{-(cm^2\pi^2/4l^2)} \sin \frac{m\pi}{2l} y. \tag{113}$$

When $\tau = 0$, $w_1 = \varphi(y)$ and therefore

$$S_m = \frac{1}{l}\int_0^l \left\{ \sum_{n=1, 3, \cdots} K_n \sin \frac{n\pi}{2h}(y + h - l) \sin \frac{m\pi}{2l} y dy \right.$$
$$\left. + \frac{1}{l}\int_l^{2l} \sum_{n=1, 3, \cdots} K_n \sin \frac{n\pi}{2h}(h + l - y) \sin \frac{m\pi}{2l} y dy \right\}.$$

Integrating we shall have:

$$S_m = \frac{4\pi}{l^2} \sum_{n=1, 3, \cdots} K_n \frac{m \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m\pi}{l}\right)^2 - \left(\frac{n\pi}{h}\right)^2}, \tag{114}$$

S_m being zero for even values of m. Hence

$$w_1 = \frac{4}{l^2} \sum_{m=1, 3, \cdots} \sum_{n=1, 3, \cdots} K_n \frac{m \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m\pi}{l}\right)^2 - \left(\frac{n\pi}{h}\right)^2}$$
$$\times e^{-(cm^2\pi^2/4l^2)\tau} \sin \frac{m\pi}{2l} y. \tag{115}$$

It is more convenient to have our equations referred to a single origin of coordinates. Take this as the bottom surface and let z be the distance of any section from this surface. We have

$$x = h - z \qquad \text{for} \qquad l \leq z \leq h,$$

$$y = l - z \qquad \text{for} \qquad 0 \leqq z \leqq l.$$

And therefore

$$w_2 = \frac{2}{\pi(h-l)^2} \sum_{m=1}^{m=\infty} \sum_{n=1,3,\cdots} K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi \cos \frac{n\pi}{2h} l}{\left(\frac{n}{2h}\right)^2 - \left(\frac{m}{h-l}\right)^2} \times e^{-(cm^2\pi^2/(h-l)^2)t} \sin \frac{m\pi}{h-l}(h-z), \quad (116)$$

for $l \leq z \leq h$, and

$$w_1 = \frac{4}{\pi l^2} \sum_{m=1,3,\cdots} \sum_{n=1,3,\cdots} K_n \frac{m \sin \frac{n\pi}{2} \cos \frac{n\pi}{2h} l}{\left(\frac{m}{l}\right)^2 - \left(\frac{n}{h}\right)^2} \times e^{-(cm^2\pi^2/4l^2)t} \sin \frac{m\pi}{2l}(l-z), \quad (117)$$

for $0 \leq z \leq l$. In both cases:

$$p_{1,2} = \gamma(h-z) - w_{1,2}.$$

3. Sand Layer at the Bottom of the Deposit. In case the layer of sand is located immediately above the bottom impermeable surface, the pressure distribution will be the same as that given by equation (116) by setting $l = 0$. It is:

$$w = \frac{8}{\pi} \sum_{m=1}^{m=\infty} \sum_{n=1,3,\cdots} K_n \frac{m \sin \frac{n\pi}{2} \cos m\pi}{n^2 - 4m^2} \times e^{-(cm^2\pi^2/h^2)t} \sin \frac{m\pi}{h}(h-z), \quad (118)$$

and $p = \gamma(h-z) - w$, the solution now holding for

$$0 \leq z \leq h.$$

4. Settlements and Rates of Settlement. The effect of drainage on the settlements and rates of settlement of the top surface of the deposit can be found by applying equations (56) and (57) to the above equations.

We have

$$\frac{ds}{d\tau} = -\int_0^l a\frac{\partial w_1}{\partial \tau}dz - \int_l^h a\frac{\partial w_2}{\partial \tau}dz. \tag{119}$$

Substituting the values of $\partial w_1/\partial\tau$ and $\partial w_2/\partial\tau$ from (116) and (117) and taking a constant during integrations, we get

$$\frac{ds}{d\tau} = \frac{2k}{l^2}\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots} K_n \frac{m^2 \sin\frac{n\pi}{2}\cos\frac{n\pi}{2h}l}{\left(\frac{m}{l}\right)^2 - \left(\frac{n}{h}\right)^2} e^{-(cm^2\pi^2/4l^2)\tau}$$

$$+ \frac{4k}{(h-l)^2}\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots} K_n \frac{m^2 \sin\frac{n\pi}{2}\cos\frac{n\pi}{2h}l}{\left(\frac{m}{h-l}\right)^2 - \left(\frac{n}{2h}\right)^2} e^{-[cm^2\pi^2/(h-l)^2]\tau}. \tag{120}$$

as the rate of settlement of the top surface of the deposit. The total settlement of the top surface at any time τ, due to drainage alone, is:

$$s = \int_0^\tau \frac{ds}{d\tau}d\tau = \frac{8a}{\pi^2 l}\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots} K_n \frac{\sin\frac{n\pi}{2}\cos\frac{n\pi}{2h}l}{\left(\frac{m}{l}\right)^2 - \left(\frac{n}{h}\right)^2}$$

$$\times (1 - e^{-(cm^2\pi^2/4l^2)\tau}) + \frac{4a}{\pi^2(h-l)}$$

$$\times \sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots} K_n \frac{\sin\frac{n\pi}{2}\cos\frac{n\pi}{2h}l}{\left(\frac{m}{h-l}\right)^2 - \left(\frac{n}{2h}\right)^2}(1 - e^{-[cm^2\pi^2/(h-l)^2]\tau}). \tag{121}$$

Equations (120) and (121) refer to the case in which the layer of sand is located at a distance l from the bottom surface.

For the case of the sand layer located at the bottom surface, we have from equation (120) by setting $l = 0$ in the second term of the right hand member,

$$\frac{ds}{d\tau} = \frac{16k}{h}\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots} K_n \frac{m^2 \sin\frac{n\pi}{2}}{4m^2 - n^2} e^{-(cm^2\pi^2/h^2)\tau}. \tag{122}$$

Placing $l = 0$ in the second term of the right hand member of equation (121) we get

$$s = \frac{16ah}{\pi^2} \sum_{m=1,3,\cdots} \sum_{n=1,3,\cdots} K_n \frac{\sin \frac{n\pi}{2}}{4m^2 - n^2} (1 - e^{-(cn^2\pi^2/h^2)\tau}). \quad (123)$$

These two last equations give the rates of settlement and total settlement of the top surface of the deposit at any time, τ.

It should be noticed that the total settlement given by equation (121) and (123) is that due to drainage alone. Therefore, if the total settlement from the beginning of the second stage, to a time $\tau = \tau_1$, is desired, we will have to add to the above equations, a constant term giving the total settlement at a time $t = t_1$ corresponding to $\tau = 0$.

5. Example. We will now determine the stress distribution and settlements of the mud deposit discussed in Problem I, for $t = 500$ and $t = 1{,}000$ years. Two cases will be considered: that in which the sand layer is located at the center of the deposit, and that in which it is located at the bottom.

The constants are:

$c = 3{,}395$, $a = 0.00024$, $h = 2{,}500$ cm., $l = 1{,}250$ cm. and $l = 0$.

(*a*) *Sand Layer at Middle of Deposit.* The results of computations are as follows:

$\frac{z}{h}$	Initial State: $t = 500$ Years			Initial State: $t = 1{,}000$ Years		
	Values of w for (gr./cm.²)			Values of w for (gr./cm.²)		
	$\tau = 50$	$\tau = 100$	$\tau = 500$	$\tau = 50$	$\tau = 100$	$\tau = 500$
0	1,370	1,060	97	700	540	50
1/4	950	750	68	485	380	35
3/8	514	405	37	260	210	27
5/8	193	64	—	96	33	—
3/4	266	91	—	136	47	—

The rates of settlement and total settlements are:

	Initial State: t = 500 Years				Initial State t = 1,000 Years			
	τ = 10	τ = 50	τ = 100	τ = 500	τ = 10	τ = 50	τ = 100	τ = 500
$\frac{ds}{d\tau}$ in cm./yr.	5.83	2.65	1.43	0.12	2.98	1.35	0.73	0.07
s in m.	0.69	2.14	3.09	5.50	0.35	1.09	1.57	2.81

(*b*) *Sand Layer at the Bottom of the Deposit*: The hydrodynamic pressures are:

	Initial State: t = 500 Years			Initial State: t = 1,000 Years		
$\frac{z}{h}$	Values of w for (gr./cm.^{2})			Values of w for (gr./cm.^{2})		
	τ = 50	τ = 100	τ = 500	τ = 50	τ = 100	τ = 500
1/8	501	330	34	256	168	17
1/4	873	590	62	445	296	32
1/2	986	750	88	503	382	45
3/4	523	470	34	262	240	17

The rates of settlement and total settlements are:

	Initial State: t = 500 Years.				Initial State: t = 1,000 Years.			
	τ = 10	τ = 50	τ = 100	τ = 500	τ = 10	τ = 50	τ = 100	τ = 500
$\frac{ds}{d\tau}$ in cm./yr.	4.26	2.02	1.53	0.14	2.18	1.03	0.79	0.07
s in m.	0.52	1.69	2.59	5.21	0.27	0.86	1.32	2.66

These results are plotted in Figs. (23), (24), (25), and (26). From them we learn how much more rapid the process of consolidation due to drainage is than that due to the weight of the material itself. Consider, for instance, the deposit whose

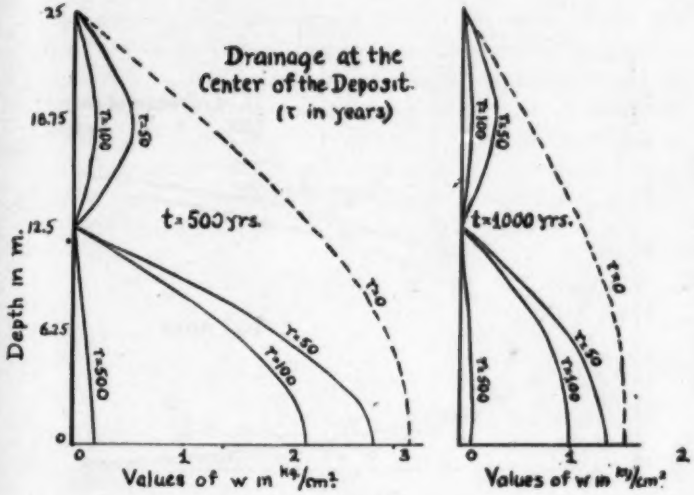
Drainage at the
Center of the Deposit.
(τ in years)
t=500 yrs.
t=1000 yrs.
τ=0
τ=50
τ=100
τ=500
25
18.75
12.5
6.25
0
Depth in m.
0
1
2
3
Values of w in kg/cm²

FIG. 23.

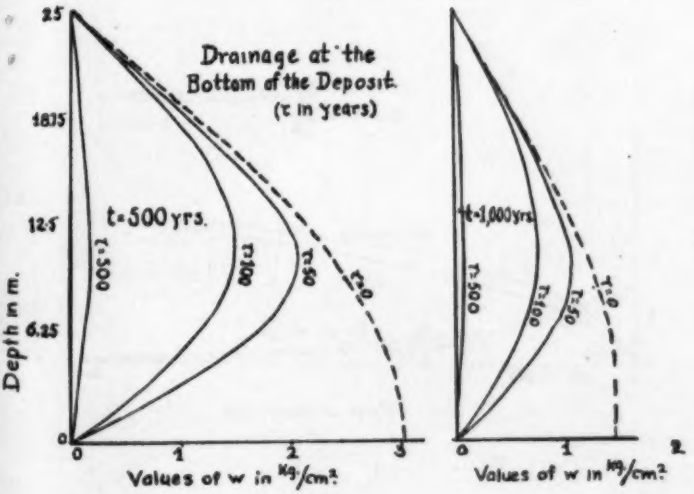
Drainage at the
Bottom of the Deposit.
(τ in years)
t=500 yrs.
t=1,000 yrs.
τ=0
τ=50
τ=100
τ=500
25
18.75
12.5
6.25
0
Depth in m.
0
1
2
3
Values of w in kg/cm²

FIG. 24.

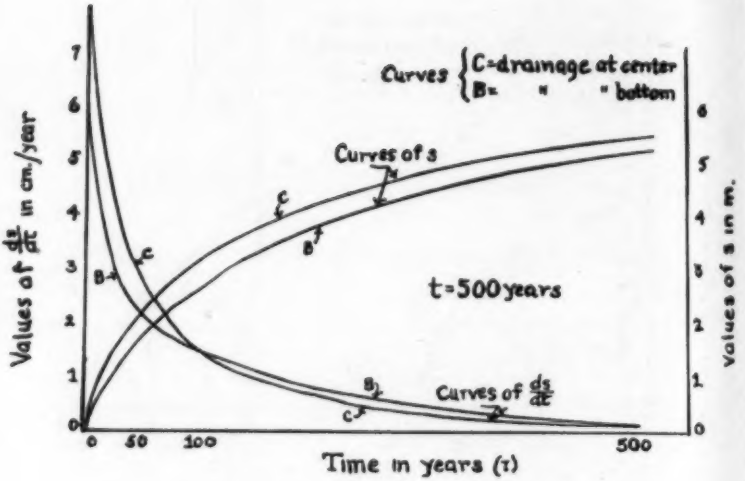
Curves { C = drainage at center
B = " " bottom
Curves of s
t = 500 years
Curves of $\frac{ds}{dt}$
Values of $\frac{ds}{dt}$ in cm/year
Values of s in m.
Time in years (t)
0
50
100
500

FIG. 25.

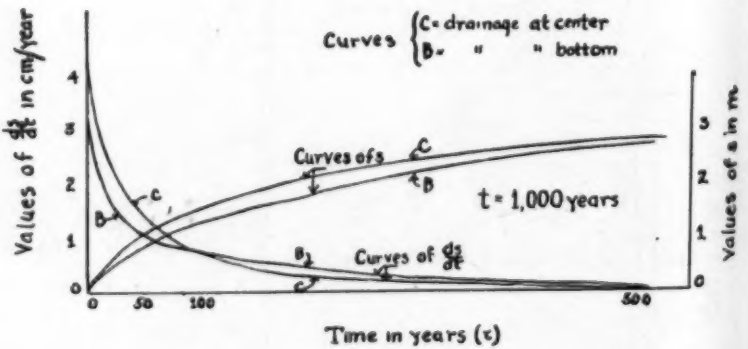
Curves { C = drainage at center
B = " " bottom
Curves of s
t = 1,000 years
Curves of $\frac{ds}{dt}$
Values of $\frac{ds}{dt}$ in cm/year
Values of s in m.
Time in years (t)
0
50
100
500

FIG. 26.

initial state ($\tau = 0$) corresponds to $t = 500$ years. After 500 years the total settlements for the two cases considered are 5.50 m. and 5.21 m. while the total settlement due to the weight of the material itself for the same period of time, ($t = 500$ to $t = 1{,}000$ years) is 3.03 m. (Problem I). This difference is not very large, but if we compare the rates of settlement due to drainage with those due to the weight of the material itself (Problem I), we will notice a very large difference.

ON THE DERIVATION OF POISSON'S DISTRIBUTION LAW

By D. J. Struik

§ 1. De Moivre, in his "Doctrine of Chances" [1] derived from the binomial distribution function (or Bernoulli law)

$$P_m(n) = C_n^m p^n (1-p)^{m-n} \tag{1.1}$$

(m and p constants, $p < 1$, m and n integers), with

$$C_n^m = \frac{m!}{n!(m-n)!},$$

a law, closely related to the case $n = 1$ of the distribution function

$$P(n) = \frac{e^{-\nu}\nu^n}{n!}, \tag{1.2}$$

in which ν is a constant. De Moivre indeed studied distributions for which p is small and m large and so connected that

$$mp = \text{finite}$$

with increasing m and decreasing p. and this is the essential reasoning by which the general law (1.2) can be derived from (1.1).

Poisson [2] found the general expression (1.2), but, just as De

[1] A. de Moivre, *Doctrine of Chance*, 1718, Problem *V*, p. 14. This was brought to my knowledge in a paper by Ethel M. Newbold: "Practical Applications of the Statistics of Repeated Events, particularly to Industrial Accidents," *Journal Roy. Statist. Society*, 90 (1927), pp. 487–535. It is curious that recently the origin of the normal law has also been traced back to De Moivre, see f.i. D. E. Smith, "Source Book in Mathematics." The relation to Poisson's law, however, is still remote in De Moivre, and, contrary to Miss Newbold, I think that we are historically justified in calling the law after Poisson.

[2] S. D. Poisson, "Recherches sur la probabilité des jugements en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilités," Paris, 1837, pp. 40 and 205. On p. 40 the relation to De Moivre's work is clear.

Moivre in his special case, as a transitory result. It is not apparent that they attached much weight to it. In the last twenty years the importance of this law for many cases has been recognized, especially after the publication of Von Bortkiewicz's little book on "the law of small numbers" (perhaps better called "the law of small probabilities"), in 1898.[3]

Von Bortkiewicz accepted the law as Poisson had found it, as a special case of the Bernoulli law, and from that point of view it is still considered by several authors. As an example, we take a paper by Pollaczek-Geiringer in 1927,[4] followed by a generalization of Poisson's law in 1928 [5] in the same line of thought. In 1910, Lucy Whittaker analyzed the use in actual statistics of Poisson's law as a limiting case of the binomial law, she showed that it often corresponds to negative m, and concluded that a general interpretation based on a simple conception was needed for such cases in which it had turned out to correspond to such a binomial law with negative m.[6]

The clever statistician of the Pearson school, who hides himself under the name of "Student," had already given, in 1905, an interpretation of Poisson's law which does not depart from the principles of Poisson's derivation, but makes its occurrence more plausible.[7] In 1918 he took up Miss Whittaker's suggestion and, on the same basis as in his 1905 paper, gave some general rules applying to those distributions under which the Poisson law holds.[8] These rules are (a large number of individuals being distributed over a large number of divisions:)

[3] L. von Bortkiewicz, "Das Gesetz der kleinen Zahlen," Leipzig, 1898.

[4] H. Pollaczek-Geiringer, "Theorie der Statistik seltener Ereignisse," *Zeitschrift für angewandte Mathematik und Mechanik*, 7 (1927), pp. 445–446.

[5] H. Pollaczek-Geiringer, "Ueber die Poissonsche Verteilung und die Entwicklung willkürlicher Verteilungen," *Ib.* 8 (1928), pp. 292–309.

[6] L. Whittaker, "The Poisson Law of Small Numbers," *Biometrika*, 10 (1910), pp. 36–71.

[7] "Student," "On the Error of Counting with a Haemacycometer," *Ib.* 5 (1905), pp. 351–360. The same ideas in *M.v. Smoluchowski*, Über Unregelmäszigkeiten in der Verteilung von Gasmolekülen, *Festschrift L. Boltzman*, 1904, p. 626–641.

[8] "Student," "An Explanation of Deviations from Poisson's Law in Practice," *Biometrika*, 12 (1918–19), pp. 211–215.

(1) That the chance of falling in a division is the same for each individual.

(2) That the fact that an individual has fallen in a division does not affect the chance of other individuals falling therein.

(3) That the chance of an individual falling in it is the same for each division. We will call these the *"Student" conditions.*

In the meantime Student's arguments of 1905 had been stated, independently, by Borel.[9] In his book he solves a large number of interesting mathematical problems on the Poisson distribution of points on a line.

The Poisson law is, in all these investigations, a limiting case of the Bernoulli law. The "Student" conditions themselves, however, do not imply this law. In 1910 Bateman,[10] ignorant of Poisson's and Bortkiewicz' work, showed how we can come to a Poisson distribution without any reference to the binomial law, by simply assuming that a distribution of points on a line obeys the following laws:

(1) The chance that a point falls in a segment dx of the line is kdx, k independent of x.

(2) The chance that $n + 1$ points fall in a segment of length $x + dx$ is the sum of the chances that

(*a*) No points fall in segment x and no point falls in segment dx, and

(*b*) n points fall in segment x and 1 point falls in segment dx. (Bateman does not take condition (2) as a special assumption, but uses it as self-evident). We will call these conditions the *Bateman conditions.*

In such manner we see connection with the investigations by Charlier, which established a kind of duality between the normal

[9] E. Borel, "Introduction géométrique a quelques théories physiques," Paris, 1914, Note V, pp. 109–112. Further work on the subject in E. Borel, "Traité du Calcul des Probabilités et de ses Applications," Tome I, Fasc. I (redigées par R. Lagrange), Paris, 1924, p. 57 and following.

[10] H. Bateman, "On the Probability Distribution of a Particle," *Philos. Magazine* (6), 20 (1910), pp. 704–707. In a paper "Some problems in the theory of probabilities," *Ib.* (6) 21 (1911) p. 745–752 Bateman discusses several applications of the Poisson law in physics.

law and the Poisson law, both obtainable independently of the binomial law. In 1921 Fujiwara took up Bateman's investigations. He remarked that Bateman accepted condition (2) as self evident and showed what could be done by making a special assumption instead. He did not take (2), but the more general condition which Fry recently called the "collectively at random" condition. Fry [11] assumed that

(1) The chance that n points fall in a segment of length x depends only on n and x, not on any information about line segments not wholly or partly included in x. (This is called *collectively at random*.)

(2) Each point of the set is placed at random, independent of all the rest (this is called *individually at random*.) We will call these conditions the *Fujiwara-Fry conditions*. Fry stated condition (1) in a paper of 1925.[12]

In this paper we will compare the derivation of the Poisson distribution from the binomial law with the derivation on an independent basis, compare the different conditions, and study one of the generalizations.

§ 2. In the derivation from the binomial law, as done by "Student" in 1905 and by Borel, we will first suppose that N *congruent balls are to be distributed among l urns, each of which is as likely as the others*. Be Px the probability that one ball comes in one of x of the l urns. Then the probability that n balls are in x of the l urns is given by the binomial law:

$$p(n,x) = \frac{N!}{n!(N-n)!} p_x^n (1-p_x)^{N-n}. \tag{2.1}$$

Now change the statement of the problem as follows: *Distribute N points over a line segment of length l, each element of which is as likely as the other. Required be the probability that n points are in a segment of length x.*

[11] M. Fujiwara. Ein Problem der Wahrscheinlichkeitsrechnung, Tôhoku Mathematical Journal 20(1921) p. 40–50; T. C. Fry, "Probability and Its Engineering Uses," New York, Van Nostrand, 1928, p. 232.

[12] T. C. Fry, "The theory of the Schroteffekt," *Journal Franklin Institute*, 199 (1925), pp. 203–220.

The probability p_x that a point comes in a segment of length x is $\frac{x}{l}$. Therefore the required probability is

$$p(n,x) = \frac{N!}{n!(N-n)!}\left(\frac{x}{l}\right)^n\left(1-\frac{x}{l}\right)^{N-n}. \tag{2.2}$$

Let us now suppose that the line segment l extends to infinity and that at the same time the number of points N increases indefinitely. In this case we find for

$$p(0,x) = \left(1-\frac{x}{l}\right)^N, \tag{2.3}$$

then and only then, a finite limit, different from 0 or 1, if

$$\lim_{\substack{N\to\infty \\ l\to\infty}} \frac{N}{l} = \text{finite, say } \delta, \tag{2.4}$$

where δ is a positive number differing from zero. We then find for $p(0, x)$ in the limit

$$p(0,x) = \lim_{l\to\infty}\left(1-\frac{x}{l}\right)^{\delta l} = l^{-\delta x}. \tag{2.5}$$

From (2.2) we get

$$\frac{p(n+1,x)}{p(n,x)} = \frac{N-n}{n+1}\,\frac{x}{l-x} = \frac{x}{n+1}\,\frac{N-n}{l-x}, \tag{2.6}$$

which in the limit passes into:

$$\frac{p(n+1,x)}{p(n,x)} = \frac{\delta x}{n+1}. \tag{2.7}$$

From (2.7) and (2.5) we find by passing from $p(0, x)$ to $p(1, x)$, from $p(1, x)$ to $p(2, x)$, etc.,

$$p(n,x) = \frac{e^{-\delta x}(\delta x)^n}{n!}, \tag{2.8}$$

which is Poisson's law (1.2) for $\nu = \delta x$.

The expected number of balls in x urns is, according to (2.1)

$$\epsilon(x) = \sum np(n,x) = Npx. \tag{2.9}$$

This passes into

$$\epsilon(x) = N\frac{x}{l}, \tag{2.10}$$

for the expected number of points on a segment of length x, and by virtue of (2.4) into

$$\epsilon(x) = \delta x. \tag{2.11}$$

the expectation of Poisson's law.

In the same way, we find for the standard deviation for (2.1)

$$\sigma = \sqrt{\sum (n - Np_x)^2 p(n, x)} = Np_x(1 - p_x),$$

which passes into

$$\sigma = \sqrt{N\frac{x}{l}\left(1 - \frac{x}{l}\right)}$$

for (2.2), and into

$$\sigma = \sqrt{\epsilon}$$

for the Poisson law. This gives $\epsilon = \sigma^2$ for the Poisson law.

§ 3. Let us now set up a series of conditions containing the "Student," Bateman and Fujiwara-Fry conditions, which lead directly to the Poisson law without passing to a limit in the binomial law. For this, we introduce for a moment a denumerable number of points of different weight $P_1, P_2, \cdots, P_k \cdots$, and call $p(P_{i1}, P_{i2}, \cdots P_{in}, l, x)$ the probability that the points $P_{i1}, P_{i2}, \cdots P_{in}$, fall on the line segment of length l between l and $l + x$. In this way of writing it is already assumed that there are no other quantities than the P, l and x to determine the chance. (We might f.i., suppose that the distribution of the other points influences that of the $P_{i1} \cdots P_{in}$). Then we assume.

(1) All points have the same weight, so that they may be taken as congruent. This reduces the probability function to $p(n, l, x)$ as the chance that n points fall on the segment between x and $x + l$.

(2) $p(n, l, x)$ is independent of l, so that we may write $p(n, x)$.

(3) $\lim_{x \to \infty} \frac{p(1, x)}{x} = k$, a constant.

(4) $\lim_{x \to \infty} \frac{p(i, x)}{x} = 0, \quad i = 2, 3, 4, \cdots.$

The Student conditions 1, 2, 3 return as 1, 2, 4; the Bateman condition 1 returns as 3, the other is, as we shall see, a consequence of 2, 3. 4; the Fujiwara-Fry conditions 1 and 2 are our conditions 2 and 4.

The derivation of Poisson's law from these conditions is very simple, and goes entirely as Bateman has done.

From (2) we have

$$p(n, x + h) = p(n, x)p(0, h) + p(n - 1, x)p(1, h) + \cdots + p(0, x)p(n, h),$$

and, as

$$p(0, h) = 1 - p(1, h) - p(2, h) - \cdots - p(n, h) - \cdots$$

we find:

$$(3.5) \quad \frac{p(n, x + h) - p(n, x)}{h} = [- p(n\ x) + p(n - 1, x)] \frac{p(1, h)}{h} + [- p(n, x) + p(n - 2, x)] \frac{p(2, h)}{h} + \cdots.$$

But according to (3) we have

$$\lim_{h \to 0} \frac{p(1, h)}{h} = k,$$

so that, according to (4), $\frac{dp(n, x)}{dx}$ exists, and is equal to

$$\frac{dp(n, x)}{dx} = k[- p(n, x) + p(n - 1, x],$$

equation given by Bateman, who, however, assumed (4) as self-evident, contrary to Fry later. The first equation of this set is:

$$\frac{dp(0, x)}{dx} = - kp(0, x).$$

We know besides $p(0, 0) = 1$, $p(n, 0) = 0$ and this gives as

solution, by successive steps:

$$p(n, x) = \frac{(kx)^n e^{-kx}}{n!},$$

the Poisson law.

Fujiwara has shown, besides, that from condition (2) alone, that is, from equation (3.5), we can find an explicit expression for $p(n, x)$, which passes into the Poisson expression through condition (3).

§ 4. We can also find the Poisson law by substituting for assumption (3) the other

(3^a) $p(0, x)$ is finite and different from 0 and 1.

Then the derivation, however, lasts somewhat longer. This condition is implicit in Fry.

Then we follow, from (1), for a segment of finite length x

$$p(0, x) = p\left(0, \frac{x}{n}\right)^n. \tag{4.1}$$

But as

$$p\left(0, \frac{x}{n}\right) = 1 - H, \qquad H = \sum_{i=1}^{\infty} p\left(i, \frac{x}{n}\right),$$

so that

$$p(0, x) = (1 - H)^n,$$

we must have, for $n \to \infty$, and a well known law of the theory of infinite products:

$$\lim_{n \to \infty} Hn = \text{finite and different from zero, say} = k(x).$$

Hence:

$$p(0, x) = e^{-k(x)},$$

and, from (4.1)

$$e^{-k(x)} = e^{-nk(x/n)}.$$

The equation (see f.i. E. Picard's "Equations fonctionelles")

$$k(\varphi) = \frac{1}{n} k(n\varphi), \tag{4.2}$$

where n runs from 0 to ∞ over all positive values, admits only

one continuous solution existing for all real values of φ, namely

$$k(\varphi) = k\varphi, \tag{4.3}$$

where k is a constant:

$$k = k(1).$$

This can be seen by deducing from (4.2) successively

$$k\left(\frac{n\varphi}{p}\right) = nk\left(\frac{\varphi}{p}\right) \qquad \text{(for any real } p\text{)},$$

$$k(\varphi) = pk\left(\frac{\varphi}{p}\right) \qquad \text{(for } n = p\text{)},$$

$$\frac{n}{p}k(\varphi) = k\left(\frac{n\varphi}{p}\right) \qquad \text{(by division)},$$

$$k\left(\frac{n}{p}\right) = \frac{n}{p}k(1) \qquad \text{(for } \varphi = 1\text{)}.$$

The equation is now proved for rational values of φ, and therefore for all positive values, if $k(\varphi)$ is assumed to be continuous. Hence:

$$p(0, x) = e^{-kx}.$$

This leads immediately, using assumption (4), to

$$\lim_{x \to 0} \frac{p(1, x)}{x} = \lim_{x \to 0} \frac{1 - e^{-kx}}{x} = k,$$

which is assumption (3).

§ 5. In the binomial law assumption (4), as we saw, corresponds to the balls being congruent, (2) to the urns being equally likely, (3) and (4) to the fact that each ball falls independent of the other, also in the limit. Then (2.4) corresponds to (3).

§ 6. Among the generalizations that we can make, we will consider the case that (2) is omitted. Fry shows how this can be found without the binomial law. We will here show how this distribution can also be obtained from a modified binomial law, with variable probabilities. For this purpose we therefore change from a Bernoulli scheme to a Poisson scheme, the expression

"Poisson scheme" used as is done in statistics (opposed to Bernoulli and Lexis scheme, without any relation to the conception of "Poisson's distribution law").

Suppose that N points have to be distributed over a line-segment l of which not all segments are equally likely. In the expression for the probability the l enters as a parameter.

Let $p(x, l)dx$ now be the probability that a point on a line of length l comes in the segment between the points x and $x + dx$. Then the probability that n points lie in the segment between the points x and $x + h$ is:

$$p(n, x, h) = \frac{N!}{n!(N-n)!} p_{x,h}^{n}(1 - p_{x,h})^{N-n}, \tag{6.1}$$

where

$$p_{x,h} = \int_{x}^{x+h} p(x, l)dx$$

and on account of this

$$p(0, x, h) = (1 - p_{x,h})^{N}. \tag{6.2}$$

Now let N and l become infinite, under the condition, that

$$\lim_{\substack{N\to\infty \\ l\to\infty}} Np(x, l) = p(x)$$

is a function lying between 0 and 1 for all values of x, and such that its integral from x to $x + h$

$$N \lim_{\substack{N\to\infty \\ l\to\infty}} \int_{x}^{x+h} p(x, l)dx = \int_{x}^{x+h} p(x)dx = \text{(say) } k(x, h)$$

exists. Then (6.2) passes into

$$p(0, x, h) = e^{-k(x,h)}.$$

According to (6.1)

$$\frac{p(n+1, x, h)}{p(n, x, h)} = \frac{N-n}{n+1} \frac{p_{x,h}}{1 - p_{x,h}}.$$

In the limit this leads to

$$\frac{p(n+1, x, h)}{p(n, x, h)} = \frac{k(x, h)}{n+1},$$

so that our final result is

$$p(n, x, h) = \frac{e^{-k(x,h)}[k(x, h)]^n}{n!},$$

which has a form similar to that of the Poisson formula. For $k(x, h)$ we may write

$$k(x, h) = \int_0^h p(x + h)dh.$$

Thus we can reach, by a way similar to the "Student"-Borel method, a generalization of Poisson's law. This is worked out in another way by Fry.[13] In the same line of thought is a generalization by Greenwood and Yule.[14]

[13] Fry, l. c. (11), p. 235.

[14] Major Greenwood and G. U. Yule, "An Inquiry Into the Nature of Frequency Distributions," *Journal Roy. Statistical Soc.*, 83 (1920), pp. 255–279.

MUTUAL INDUCTANCE OF SHORT CONCENTRIC SOLENOIDS

H. B. Dwight and P. W. Sayles

In comparing formulas for the self-inductance of solenoids, it is noticed that a series which is most rapidly convergent for very short solenoids, namely that of Rayleigh and Niven,[1] contains log terms, while most series formulas which are convergent for very long solenoids, such as the Webster-Havelock formula,[2] do not contain log terms. Similarly, the series formula of Maxwell [3] for the mutual inductance of two circles near together contains log terms, but many series formulas for the mutual inductance of two circles which are far apart, such as eq. (5) of Scientific Paper 169 of the Bureau of Standards, do not contain log terms.

It would be expected from the preceding paragraph that a series formula containing log terms would be very rapidly convergent for mutual inductance of short solenoids close together. This has been found to be the case. Such a formula has been derived and is presented in this article. The range of application is indicated in Fig. 2, and it is shown that for very short solenoids near together, the other series formulas with which comparison is made, are not rapidly enough convergent for practical numerical calculations.

The mutual inductance, in abhenrys, of two short, thin, coaxial and concentric solenoids of N_1 and N_2 turns, of diameters m and $m + 2p$ cm., respectively, and of equal length b cm., (See Fig. 1), is

$$M = \pi m N_1 N_2 \left[\left(\log_n \frac{16m^2}{b^2 + p^2} \right) \left(1 + \frac{p}{m} + \frac{b^2}{8m^2} + \frac{p^2}{4m^2} \right. \right.$$

[1] *Proc. Royal Soc.*, Vol. 32, p. 104, 1881. See also formulas (69) and (71) *Sci. Pa. 169 of the Bur. of Standards.*

[2] *Bull. of Amer. Math. Soc.*, Vol. 14, No. 1, p. 1, 1907; *Phil. Mag.*, Vol. 15, p. 332, 1908; and eq. (79), *Sci. Pa. 169 of the Bur. of Standards.*

[3] *Elec. and Mag.*, Para. 705 and eq. (10), *Sci. Pa. 169 of the Bur. of Standards.*

$$-\frac{b^2p}{8m^3}-\frac{p^3}{4m^2}-\frac{b^4}{64m^4}+\frac{7}{64}\frac{b^2p^2}{m^4}+\frac{17}{64}\frac{p^4}{m^4}+\cdots\Bigg)$$
$$+\frac{p^2}{b^2}\left(\log_n\frac{b^2+p^2}{p^2}\right)\left(1+\frac{p}{m}-\frac{p^2}{8m^2}+\frac{p^3}{8m^3}-\frac{9}{64}\frac{p^4}{m^4}\right.$$
$$\left.+\cdots\right)-4\frac{p}{b}\left(1+\frac{p}{m}\right)\tan^{-1}\frac{b}{p}-1+\frac{p}{m}+\frac{b^2}{16m^2}$$
$$+\frac{13}{8}\frac{p^2}{m^2}+\frac{3}{16}\frac{b^2p}{m^3}-\frac{11}{24}\frac{p^3}{m^3}+\frac{b^4}{48m^4}$$
$$-\frac{5}{16}\frac{b^2p^2}{m^4}+\frac{p^4}{24m^4}+\cdots\Bigg]. \quad (1)$$

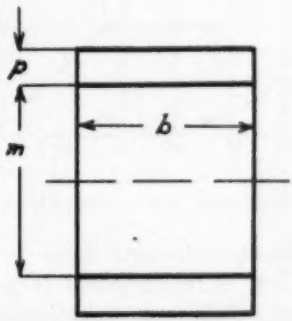

FIG. 1.—Two Coaxial, Concentric Solenoids of Equal Length.

This is derived by integrating eq. (14) of Scientific Paper 169, which is an extension of the formula of Reference 3, twice over the length of the solenoids. Note that log$_n$ denotes natural logarithm.

The mutual inductance of two short solenoids of unequal length is obtained in a manner similar to the derivation of (1). As the formula is somewhat long, it is not given here. It may be found in the S. M. thesis by P. W. Sayles, Mass. Institute of Technology, 1929, equation (16).

Formula (1) gives the mutual inductance of two thin solenoids with slide-rule accuracy, for values of p/m up to .25 and of b/m up to .8, as is indicated by the vertical dotted line of Fig. 2.

The convergence is most rapid for very small values of b/m. Other series formulas which apply to this case have the most rapid convergence for large values of b/m, and as b/m decreases, they give less accurate results and finally cease to be of practical use. A comparison of several formulas for coils of equal length is given on page 75 of Scientific Paper 169 of the Bureau of

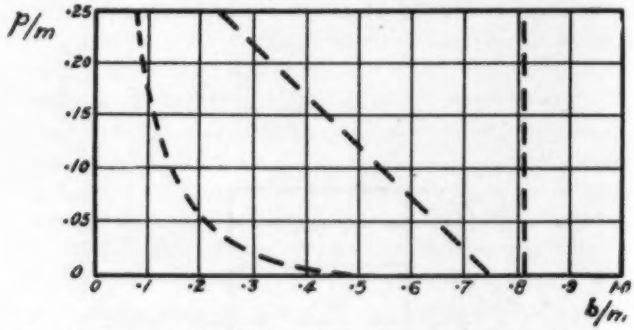

FIG. 2.—Ranges of Application of Two Mutual Inductance Formulas.

Standards, and Maxwell's formula given in Electricity and Magnetism, para. 678, is recommended in preference to the others for short coils. This formula, considerably extended, is given in Eq. (36), Scientific Paper 169.

A comparison of the applicability of Formula (1) of this paper with that of Eq. (36) mentioned in the preceding paragraph, is given in Fig. 2. It is seen that there is a considerable range in which both formulas can be used, but that for cases represented by the region to the left of the curved line, Eq. (1) is to be used, if a series formula is employed as is usual for computations of this kind. While the curved line of Fig. 2 shows the limit beyond which Maxwell's formula in its extended form cannot be used, the sloping dotted line shows the approximate boundary to the left of which Maxwell's formula is inferior to, that is, more slowly convergent than, Eq. (1) of this paper.

A measurement of the mutual inductance of two concentric

solenoids of equal length was made. The length of the solenoids was 4.29 cm. The smaller coil had 95 turns and a mean diameter of 7.68 cm. The larger one had 91 turns and a mean diameter of 11.20 cm. The measured mutual inductance was .437 milhenrys and the calculated value by Eq. (1) was .450 milhenrys.

The work described was done in the Department of Electrical Engineering at the Massachusetts Institute of Technology.

TABLES AND METHODS OF EXTENDING TABLES FOR INTERPOLATION WITHOUT DIFFERENCES

By George Rutledge and Prescott Crout

1. Introduction. The classical formulas of Newton, Gauss, and Stirling approximate a function $f(x)$ by means of the algebraic polynomial $P_{2n}(x)$ of lowest degree which takes given functional values at the points $x = -nh, \cdots, -h, 0, h, \cdots, nh$, this polynomial being expressed in terms of differences of the given values. It is thus in every case necessary to form a table of differences before using such approximation formulas. On the other hand the formula of Lagrange approximates the function $f(x)$ by means of this same polynomial; but since in this case the polynomial is expressed linearly in terms of the given values, no preliminary differencing is required, and the computation of an interpolated value of $f(x)$ with a modern computing machine reduces to one continuous machine operation.

Tables of Lagrangean coefficients have been recently published by Huntington [1] which permit the insertion of ninety-nine interpolated values between each two of four or of six equally spaced points, the approximation being by means of polynomials of the third or of the fifth degree.

In this paper tables of Lagrangean coefficients are given for the cases of three, five, and seven equally spaced points. In computing these coefficients, methods and tables were used which were developed by Rutledge,[2] by means of which the Lagrangean coefficients may be computed for odd numbers of equally spaced points from three to seventeen inclusive. The computed coef-

[1] Huntington: "Tables of Lagrangean Coefficients for Interpolating Without Differences," *Proceedings of the American Academy of Arts and Sciences*, vol. 63, p. 421.

[2] Rutledge: "Fundamental Table for Lagrangean Coefficients," this *Journal*, vol. 8, p. 1.

ficients were then checked by substitution in a set of equations derived by Crout;[2] and since this set of equations constitutes not only a necessary condition, but a sufficient condition for their determination, the check is conclusive. The values given in the accompanying tables are exact terminated decimals.

Values of the error coefficient, which arises in a certain expression limiting the error,[3] are also given in this paper. These values were first computed using the newly calculated Lagrangean coefficients, the method being given by Crout in a previous paper.[3] They were then computed directly as products, which checked not only the error coefficients, but the Lagrangean coefficients as well.

In most practical problems the functional values are given only approximately; for example, the given values may be empirical, or correct to a certain number of decimal places. Methods and tables are given in this paper by means of which the additional error arising from this source may be limited.

Methods are here given by means of which the given tables of Lagrangean coefficients may be used for interpolation at points not tabulated directly, and for extrapolation; also, methods are developed whereby, by the use of matrix multiplication, the given tables may be extended indefinitely to reduce the tabular intervals or to increase the range of tabulation.

2. Lagrangean Interpolation Coefficients. Let there be given the values of a function $f(x)$ at $2n + 1$ equally spaced points. Without loss of generality the origin may be placed at the mid-point; thus the given values are

$$f(-nh), \cdots, f(-h), f(0), f(h), \cdots, f(nh), \tag{1}$$

where h is the interval between adjacent points. It may be shown that the algebraic polynomial of degree $2n$ (at most) determined by the values (1) is given by the expression

$$P_{2n}(x) = K_{-n}f(-nh) + \cdots + K_{-1}f(-h) + K_0 f(0) + K_1 f(h) + \cdots + K_n f(nh), \tag{2}$$

[2] Crout: "A Method for Deriving Formulas of Interpolation," this *Journal*, vol. 8, p. 18.

where the coefficients $K_{-n}, \cdots, K_n$ are algebraic polynomials in (x/h). Denoting (x/h) by λ, these coefficients are tabulated for $n = 1, 2, 3, \lambda = -(n+2), -(n+1), -.9, \cdots, -.1, .1, \cdots, .9, (n+1), (n+2)$ in the following tables:

TABLES OF EXACT VALUES OF LAGRANGEAN COEFFICIENTS AND OF CORRESPONDING ERROR FACTORS

	Three Points, $n = 1$, $k = \frac{1}{6}\lambda(\lambda^2 - 1)$.				
λ	K_{-1}	K_0	K_1	k	
.1	− .045	.990	.055	−.0165	− .1
.2	− .080	.960	.120	−.0320	− .2
.3	− .105	.910	.195	−.0455	− .3
.4	− .120	.840	.280	−.0560	− .4
.5	− .125	.750	.375	−.0625	− .5
.6	− .120	.640	.480	−.0640	− .6
.7	− .105	.510	.595	−.0595	− .7
.8	− .080	.360	.720	−.0480	− .8
.9	− .045	.190	.855	−.0285	− .9
2.	1.000	−3.000	3.000		−2.
3.	3.000	−8.000	6.000		−3.
	K_1	K_0	K_{-1}	$-k$	λ

If $f(x)$ has a derivative of order $2n+1$ at each point of the interval $-nh < x < nh$, it is well known that [3, 4]

$$f(x) = P_{2n}(x) + \epsilon, \qquad -nh < x < nh, \tag{3}$$

where

$$\epsilon = kh^{2n+1}f^{2n+1}(\xi), \qquad -nh < \xi < nh, \tag{4}$$

and where k is an algebraic polynomial in (x/h). The values of k

Five Points, $n = 2$, $k = \frac{1}{120}\lambda(\lambda^2 - 1)(\lambda^2 - 4)$.

λ	K_{-2}	K_{-1}	K_0	K_1	K_2	k	
.1	.00783 75	− .05985 00	.98752 50	.07315 00	− .00866 25	.00329 175	− .1
.2	.01440 00	− .10560 00	.95040 00	.15840 00	− .01760 00	.00633 600	− .2
.3	.01933 75	− .13685 00	.88952 50	.25415 00	− .02616 25	.00889 525	− .3
.4	.02240 00	− .15360 00	.80640 00	.35840 00	− .03360 00	.01075 200	− .4
.5	.02343 75	− .15625 00	.70312 50	.46875 00	− .03906 25	.01171 875	− .5
.6	.02240 00	− .14560 00	.58240 00	.58240 00	− .04160 00	.01164 800	− .6
.7	.01933 75	− .12285 00	.44752 50	.69615 00	− .04016 25	.01044 225	− .7
.8	.01440 00	− .08960 00	.30240 00	.80640 00	− .03360 00	.00806 400	− .8
.9	.00783 75	− .04785 00	.15152 50	.90915 00	− .02066 25	.00454 575	− .9
3.	1.00000 00	− 5.00000 00	10.00000 00	−10.00000 00	5.00000 00		−3.
4.	5.00000 00	−24.00000 00	45.00000 00	−40.00000 00	15.00000 00		−4.
	K_2	K_1	K_0	K_{-1}	K_{-2}	$-k$	λ

Seven Points, $n = 3$, $k = \frac{1}{5040}\lambda(\lambda^2 - 1)(\lambda^2 - 4)(\lambda^2 - 9)$.

λ	K_{-3}	K_{-2}	K_{-1}	K_0	K_1	K_2	K_3	k	
.1	−.00159 10125	.01409 18250	−.06725 64375	.98642 77500	.08220 23125	−.01557 51750	.00170 07375	−.00070 45912 5	−.1
.2	−.00295 68000	.02580 48000	−.11827 20000	.94617 60000	.17740 80000	−.03153 92000	.00337 92000	−.00135 16800 0	−.2
.3	−.00400 28625	.03445 94250	−.15241 66875	.88062 97500	.28305 95625	−.04662 15750	.00489 23875	−.00188 70637 5	−.3
.4	−.00465 92000	.03960 32000	−.16972 80000	.79206 40000	.39603 20000	−.05940 48000	.00609 28000	−.00226 30400 0	−.4
.5	−.00488 28125	.04101 56250	−.17089 84375	.68359 37500	.51269 53125	−.06835 93750	.00683 59375	−.00244 14062 5	−.5
.6	−.00465 92000	.03870 72000	−.15724 80000	.55910 40000	.62899 20000	−.07188 48000	.00698 88000	−.00239 61600 0	−.6
.7	−.00400 28625	.03291 24250	−.13068 16875	.42315 97500	.74052 95625	−.06835 65750	.00643 93875	−.00211 57987 5	−.7
.8	−.00295 68000	.02407 68000	−.09363 20000	.28089 60000	.84268 80000	−.05617 92000	.00510 72000	−.00160 51200 0	−.8
.9	−.00159 10125	.01283 78250	−.04898 64375	.13788 77500	.93074 23125	−.03384 51750	.00295 47375	−.00088 64212 5	−.9
4.	1.00000 00000	−7.00000 00000	21.00000 00000	−35.00000 00000	35.00000 00000	−21.00000 00000	7.00000 00000		−4.
5.	7.00000 00000	−48.00000 00000	140.00000 00000	−224.00000 00000	210.00000 00000	−112.00000 00000	28.00000 00000		−5.
	K_3	K_2	K_1	K_0	K_{-1}	K_{-2}	K_{-3}	$-k$	λ

for $n = 1, 2, 3, \lambda = -.9, \cdots, -.1, .1, \cdots, .9$ are tabulated in the above tables. From (4) it follows directly that

$$|\epsilon| \leq |k| h^{2n+1} M, \tag{5}$$

where M is the largest $|f^{2n+1}(x)|$ in the interval [4] $-nh < x < nh$.

In practice the functional values (1) are generally given only approximately (to a certain number of decimal places for example); thus we have an additional error ϵ' due to this cause. Let us suppose that the absolute values of the differences between the functional values and their given values are smaller than $\delta_i \geq 0$, respectively, $i = -n, \cdots, n$; then ϵ' is obviously limited by the expression

$$|\epsilon'| < \sum_{i=-n}^{n} |K_i| \delta_i. \tag{6}$$

If, in addition, we know that [5] $\delta_i < \delta$, $i = -n, \cdots, n$; then from (6) it follows that

$$|\epsilon'| < \delta \sum_{i=-n}^{n} |K_i|. \tag{7}$$

In the table on the following page the exact values of $\sum_{i=-n}^{n} |K_i|$ are given for $n = 1, 2, 3$, $|\lambda| = .1, .2, \cdots, .9$.

We now consider the following examples:

Example 1. Let us approximate sin .808 from the given values

sin .785 = .70682 51811 053
sin .795 = .71386 36049 350
sin .805 = .72083 06429 990
sin .815 = .72772 55985 995
sin .825 = .73454 77822 465,

[4] Equations (3), (4), and (5) hold also for the case where $|x| > nh$, except that in this case ξ lies somewhere within the interval bounded by the two outermost points of the set $-nh$, nh, x, and that M denotes the greatest $|f^{2n+1}(x)|$ in this same interval. For any λ, k is given by the expression

$$k = \frac{\lambda}{2n+1!} \prod_{i=1}^{n} (\lambda^2 - i^2), \qquad \lambda = \left(\frac{x}{h}\right).$$

[5] If the functional values are taken from a μ place table, then it is evident that $\delta = 10^{-\mu}$, or $\delta = 5 \cdot 10^{-\mu-1}$, depending upon the significance of the last place.

$\lvert\lambda\rvert$	$n = 1$	$n = 2$	$n = 3$
.1	1.09	1.13702 5	1.16884 525
.2	1.16	1.24640 0	1.30553 600
.3	1.21	1.32602 5	1.40608 225
.4	1.24	1.37440 0	1.46758 400
.5	1.25	1.39062 5	1.48828 125
.6	1.24	1.37440 0	1.46758 400
.7	1.21	1.32602 5	1.40608 225
.8	1.16	1.24640 0	1.30553 600
.9	1.09	1.13702 5	1.16884 525

which are correct to 13 decimal places ($\delta = 10^{-13}$). Using the five point table, we have $n = 2$, $h = .01$, $f(-2h) = \sin .785$, $\cdots, f(2h) = \sin .825, \lambda = .3$, $M = \cos .785 < .71$, $k = .00889\,525 < .0089$, $\sum_{i=-2}^{2} |K_i| = 1.32602\ 5 < 1.33$. From (5) and (7) we have, respectively,

$$|\epsilon| < .0089 \cdot 10^{-10} \cdot .71 < 6.32 \cdot 10^{-13},$$
$$|\epsilon'| < 1.33 \cdot 10^{-13};$$

hence

$$|\epsilon| + |\epsilon'| < 10^{-12}.$$

We thus see that the computed value of sin .808 should be correct to at least 12 decimal places.[6] By actual calculation this computed value is found to be (exactly)

$$P_4(.808) = .72290\ 67298\ 48993\ 3325,$$

[6] In this paper a result is *defined* to be correct, or accurate, to μ decimal places if it is known to be in error by less than $10^{-\mu}$. This does not necessarily mean that all digits to the left of and including that in the μth decimal place are identical with those of the value approximated. For example

$$1.000000\cdots$$

and

$$.999999\cdots$$

are the same number, though none of the corresponding digits coincide.

which agrees with the above prediction, for

$$\sin .808 = .72290\ 67298\ 49704 \cdots.$$

Example 2. Values of $\sin x$ are to be given at intervals of .01 along the x axis over the interval $(0, 2\pi)$. These values are to be used in connection with a five point table to compute values of $\sin x$ at intervals of .001 over this same interval, $(0, 2\pi)$. To how many decimal places should the given values be obtained; and to how many decimal places will the computed values of $\sin x$ be accurate? [6]

We have $n = 2$, $h = .01$, $|\lambda| < 1.$, $M < 1.$, $|k| < .012$, $\sum_{i=-2}^{2} |K_i| < 1.4$, $\delta = 10^{-\mu}$, where μ is to be determined. From (5) we have at once

$$|\epsilon| < .012 \cdot 10^{-10} \cdot 1 = 1.2 \cdot 10^{-12};$$

also, since from (7) we see that

$$|\epsilon'| < 1.4 \cdot 10^{-\mu},$$

it is evident that the bound upon the sum $(|\epsilon| + |\epsilon'|)$ can be reduced considerably by increasing μ, until $\mu = 12$, in which case

$$|\epsilon| + |\epsilon'| < 10^{-11};$$

hence since

$$|\epsilon| + |\epsilon'| \geq |\epsilon|,$$

and since $|\epsilon|$ is known only to be smaller than $1.2 \cdot 10^{-12}$, we conclude the following: the greatest known accuracy of the computed values of $\sin x$ is to 11 decimal places,[6] which may be obtained by using given values of $\sin x$ correct to 12 places.[7]

[7] **Reduction in the Limit upon the Error.** If $f(x)$ is such that

$$L_1 \leq f^{(2n+1)}(x) \leq L_2, \qquad -nh < x < nh; \tag{1'}$$

then from (4) we see that

$$e_1 \leq \epsilon \leq e_2,$$

where $e_1 = kh^{2n+1}L_1$ and $e_2 = kh^{2n+1}L_2$. Placing

$$\bar{e} = \frac{e_1 + e_2}{2}, \tag{2'}$$

3. Interpolation at Points Not Tabulated Directly, and Extrapolation. By means of the above tables we may compute $P_{2n}(x)$ directly for values of λ spaced at intervals of .1 throughout the two central intervals. We shall now show first how to extend the range of λ indefinitely, second how to reduce the tabular

it follows from (3) that

$$f(x) = \{P_{2n}(x) + \bar{\epsilon}\} + \epsilon'', \tag{3'}$$

where

$$|\epsilon''| \leq \frac{\epsilon_2 - \epsilon_1}{2}. \tag{4'}$$

Since $\frac{1}{2}(\epsilon_2 - \epsilon_1)$ is smaller than the larger of the quantities $|\epsilon_1|$, $|\epsilon_2|$ except in the special case where $-\epsilon_1 = \epsilon_2$, in which case $\frac{1}{2}(\epsilon_2 - \epsilon_1) = \epsilon_2$, we see from (4') that the limit upon the error is in general reduced by using $\{P_{2n}(x) + \bar{\epsilon}\}$ in place of $P_{2n}(x)$ as an approximation to $f(x)$. In order to compute $\bar{\epsilon}$ it is obviously necessary to determine L_1 and L_2. The generalization of this method to the case where $|x| > nh$ is obvious.[4]

As an example let us suppose that $f(u) = \log_{10} u$, and that the midpoint of the $2n + 1$ points is u_0; then from (4) we see that the error involved in approximating the function at any point u, where $u_0 - nh < u < u_0 + nh$, is

$$\epsilon = \frac{kh^{2n+1}\overline{2n}! \log_{10} e}{\xi^{2n+1}},$$

where e is the base of the system of natural logarithms; hence

$$\epsilon_2 = \frac{kh^{2n+1}\overline{2n}! \log_{10} e}{(u_0 - nh)^{2n+1}}, \qquad \epsilon_1 = \frac{kh^{2n+1}\overline{2n}! \log_{10} e}{(u_0 + nh)^{2n+1}},$$

$$\bar{\epsilon} = \frac{kh^{2n+1}\overline{2n}! \log_{10} e}{2}\left[\frac{1}{(u_0 - nh)^{2n+1}} + \frac{1}{(u_0 + nh)^{2n+1}}\right]; \tag{5'}$$

also

$$\frac{\epsilon_2 - \epsilon_1}{2} = \frac{kh^{2n+1}\overline{2n}! \log_{10} e}{2}\left[\frac{1}{(u_0 - nh)^{2n+1}} - \frac{1}{(u_0 + nh)^{2n+1}}\right]. \tag{6'}$$

Using the law of the mean we have

$$|\epsilon''| < \frac{knh^{2n+2}\overline{2n+1}! \log_{10} e}{(u_0 - nh)^{2n+2}}, \tag{7'}$$

due to (4') and (6'). As a special case let us suppose that $u = 6.4024$, $u_0 = 6.4020$, $n = 2$, $h = .001$; then $\lambda = .4$, $k = .010752$; and

$$0 < \epsilon < \epsilon_2 < 1.05 \cdot 10.^{-20},$$

due to (5'). From (7') we also have

$$|\epsilon''| < 1.64 \cdot 10.^{-23};$$

hence we see that by using $\{P_4(x) + \bar{\epsilon}\}$ instead of $P_4(x)$ as an approximating function in this case, we reduce the limit upon the error in the ratio $1.64 \cdot 10^{-23} + 1.05 \cdot 10^{-20} < 1.57 \cdot 10^{-3}$.

intervals for λ. In other words we shall show how by means of the above tables we may compute $P_{2n}(x)$ for values of λ spaced at intervals of .01, .001, etc. over any arbitrary range.

In each of the above tables the coefficients are given for $\lambda = -(n+2), -(n+1), (n+1), (n+2)$; thus the values of $P_{2n}(x)$ may be computed at two additional points on each side of the original $2n+1$ points, the result being that we have then a set of values of $P_{2n}(x)$ at $2n+5$ equally spaced points. We may use any consecutive $2n+1$ of these $2n+5$ values to fix $P_{2n}(x)$; however, if we include values of $P_{2n}(x)$ at one or two of the additional points, the intervals for which the coefficients are given by our table for values of λ spaced at intervals of .1 are shifted a distance h or $2h$, respectively, along the x axis, or a distance 1 or 2, respectively, along the λ axis. New values of $P_{2n}(x)$ may then be computed. We may, if we desire, repeat this process any number of times, applying the table when necessary to give $P_{2n}(x)$ for new integral values of λ; hence we determine $P_{2n}(x)$ for all values of λ spaced at intervals of .1 throughout any range.

We may now choose a set of values of $P_{2n}(x)$ at $2n+1$ consecutive values of λ spaced at intervals of .1, and use the original table to obtain $P_{2n}(x)$ for values of λ spaced at intervals of .01, since the tabular difference of .1 now corresponds to a difference of .01 in (x/h). By suitably choosing the $2n+1$ points, or by using in combination the methods of the previous paragraph, it is thus evident that we may from the original table determine $P_{2n}(x)$ for all values of λ spaced at intervals of .01 over any arbitrary range. Repeating this process, we may reduce the interval between adjacent values of λ to .001; and so on. We may thus use the original table to determine $P_{2n}(x)$ for any decimal value of λ.

4. Extension of Tables. We shall now use the methods of the previous section to determine the Lagrangean coefficients for values of λ not tabulated directly, or in other words to extend a given table. Let there be given the $2n+1$ values (1) of $f(x)$, together with a table of Lagrangean coefficients (any one of the

tables of Section 2, for example) by means of which we may compute directly the values of $P_{2n}(x)$ at $2n + 1$ equally spaced points corresponding to $2n + 1$ of the tabulated values of λ, which we denote by $\lambda_{k,0}$, where $k = -n, \cdots, -1, 0, 1, \cdots, n$. If we denote the spacing of these points along the x axis by h', it is evident that since $x = \lambda h$,

$$h' = (\lambda_{k+1,0} - \lambda_{k,0})h, \qquad k = -n, \cdots, n-1. \tag{8}$$

Noting that the coefficients $K_{-n}, \cdots, K_n$ are functions of λ, we have from (2)

$$P_{2n}(\lambda_{k,0}\, h) = \sum_{j=-n}^{n} K_j(\lambda_{k,0}) f(jh), \qquad k = -n, \cdots, n. \tag{9}$$

Using these $2n + 1$ computed values of $P_{2n}(x)$, we now apply the given table to compute the values of $P_{2n}(x)$ at m points corresponding to $\lambda' = \lambda_1', \lambda_2', \cdots, \lambda_m'$, where

$$\lambda = \lambda_{0,0} + \frac{h'}{h}\lambda', \tag{10}$$

the values $\lambda' = \lambda_1', \lambda_2', \cdots, \lambda_m'$ being arguments of the given table. Placing

$$\lambda_{i,1} = \lambda_{0,0} + \frac{h'}{h}\lambda_i', \qquad i = 1, 2, \cdots, m, \tag{11}$$

we have from (2)

$$\begin{aligned} P_{2n}(\lambda_{i,1}\, h) &= \sum_{k=-n}^{n} K_k(\lambda_i') P_{2n}(\lambda_{k,0}\, h) \\ &= \sum_{k=-n}^{n} K_k(\lambda_i') \sum_{j=-n}^{n} K_j(\lambda_{k,0}) f(jh), \end{aligned}$$

or

$$P_{2n}(\lambda_{i,1}\, h) = \sum_{j=-n}^{n} f(jh) \sum_{k=-n}^{n} K_k(\lambda_i') K_j(\lambda_{k,0}), \tag{12}$$

where $i = 1, 2, \cdots, m$. It thus follows that

$$K_j(\lambda_{i,1}) = \sum_{k=-n}^{n} K_k(\lambda_i') K_j(\lambda_{k,0}), \tag{13}$$

where $i = 1, 2, \cdots, m$, and $j = -n, \cdots, n$.

From (13) we see that the values of $K_j(\lambda_{i,1})$ form a matrix which has for the element of the ith row and jth column the term-by-term product of the ith row of the $2n+1$ column matrix formed by the values of $K_k(\lambda_i')$, and the jth column of the $2n+1$ row matrix formed by the values of $K_j(\lambda_{k,0})$. This relation may be expressed by the product

$$\| K_j(\lambda_{i,1}) \| = \| K_k(\lambda_i') \| \cdot \| K_j(\lambda_{k,0}) \|, \tag{14}$$

the multiplication so defined being the familiar row-by-column product of matrices, which is associative though not in general commutative.

It is evident that if $m = 2n+1$, and if the points corresponding to $\lambda = \lambda_{1,1}, \lambda_{2,1}, \cdots, \lambda_{m,1}$ are equally spaced, we may again apply the table as before to compute the Lagrangean coefficients at a third set of points. We thus arrive at another result of the form (14). It is evident that this process may be repeated indefinitely.[8]

Noting that for $\lambda = -n, \cdots, n$

$$\begin{aligned} K_j(i) &= 0, \quad i \neq j, \\ K_j(i) &= 1, \quad i = j, \end{aligned} \qquad i, j = -n, \cdots, n, \tag{15}$$

we see at once that the procedure of Section 3 may now be used to compute, by application of (13), (14), the Lagrangean coefficients for values of λ not tabulated directly; thus we may extend a given table indefinitely so as to increase the range of λ or to reduce the tabular intervals of λ.

To illustrate the above methods we now consider a few examples.

Example 1. Let us compute the Lagrangean coefficients for

[8] It is evident that the repetition of this process using the same table leads to a product of matrices all but the first of which are square. Similarly, if the same process is repeated using tables with different numbers of Lagrangean coefficients (different values of n), the result is a product of rectangular matrices. Such products are *associative;* also, it is evident that this property has an interesting interpretation when applied to approximating polynomials.

$n = 2$ (five points), $\lambda = 1.5$. From (14) we have

$$\begin{aligned} ||K_j(\lambda_{i,1})|| &= ||.02343\ 75 \quad -.15625\ 00 \\ &\qquad .70312\ 50 \quad .46875\ 00 \quad -.03906\ 25|| \\ &\qquad \cdot \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -5 & 10 & -10 & 5 \end{Vmatrix} \\ &= || -.03906\ 25 \quad .21875\ 00 \quad -.54687\ 50 \\ &\qquad\qquad 1.09375\ 00 \quad .27343\ 75||. \end{aligned} \tag{16}$$

Here the first matrix, $||K_k(\lambda_i')||$, of the right side of (16) is taken directly from the tables; the second matrix, $||K_j(\lambda_{k,0})||$, is obtained from the tables [9] and (15).

Example 2. Let us determine the Lagrangean coefficients for $n = 1$ (three points), $\lambda = .42, .43$. We have at once

$$\begin{aligned} ||K_j(\lambda_{i,1})|| &= \begin{Vmatrix} -.080 & .960 & .120 \\ -.105 & .910 & .195 \end{Vmatrix} \\ &\qquad \cdot \begin{Vmatrix} -.105 & .910 & .195 \\ -.120 & .840 & .280 \\ -.125 & .750 & .375 \end{Vmatrix} \\ &= \begin{Vmatrix} -.12180 & .82360 & .29820 \\ -.12255 & .81510 & .30745 \end{Vmatrix}. \end{aligned} \tag{17}$$

Example 3. Let us determine the Lagrangean coefficients for

[9] By means of the tables and (15) we may construct the following table for integral values of λ:

λ	Five Points, $n = 2$.				
	K_{-2}	K_{-1}	K_0	K_1	K_2
−1.	0	1.	0	0	0
0.	0	0	1.	0	0
1.	0	0	0	1.	0
2.	0	0	0	0	1.
3.	1	−5.	10.	−10.	5.

which gives the desired matrix directly.

$n = 1$ (three points), $\lambda = 1.213, 1.215$. We have

$$\|K_j(\lambda_{i,\,3})\| = \begin{Vmatrix} -.105 & .910 & .195 \\ -.125 & .750 & .375 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 1.000 & 0 \\ -.045 & .990 & .055 \\ -.080 & .960 & .120 \end{Vmatrix} \cdot \begin{Vmatrix} -.045 & .990 & .055 \\ -.080 & .960 & .120 \\ -.105 & .910 & .195 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{Vmatrix} \tag{18}$$

$$= \begin{Vmatrix} .12918\ 45 & -.47136\ 90 & 1.34218\ 45 \\ .13061\ 25 & -.47622\ 50 & 1.34561\ 25 \end{Vmatrix}.$$

Here as in Example 1 the first three factors of the right side of (18) are taken directly from the tables, the last being obtained from the tables and (15).

Example 4. Let us insert 9 sets of Lagrangean coefficients in a three point table ($n = 1$), corresponding to values of λ spaced at intervals of .0001 between $\lambda = 5.327$ and $\lambda = 5.328$. We have

$$\|K_j(\lambda_{i,\,6})\| = \begin{Vmatrix} -.045 & .990 & .055 \\ -.080 & .960 & .120 \\ -.105 & .910 & .195 \\ -.120 & .840 & .280 \\ -.125 & .750 & .375 \\ -.120 & .640 & .480 \\ -.105 & .510 & .595 \\ -.080 & .360 & .720 \\ -.045 & .190 & .855 \end{Vmatrix} \cdot \begin{Vmatrix} -.120 & .640 & .480 \\ -.105 & .510 & .595 \\ -.080 & .360 & .720 \end{Vmatrix} \cdot \begin{Vmatrix} -.045 & .990 & .055 \\ -.080 & .960 & .120 \\ -.105 & .910 & .195 \end{Vmatrix} \tag{19}$$

$$\cdot \begin{Vmatrix} -.080 & .960 & .120 \\ -.105 & .910 & .195 \\ -.120 & .840 & .280 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{Vmatrix}$$

$$= \begin{Vmatrix} 11.52544\ 7205 & -27.37799\ 4410 & 16.85254\ 7205 \\ 11.52592\ 9920 & -27.37905\ 9840 & 16.85312\ 9920 \\ 11.52641\ 2645 & -27.38012\ 5290 & 16.85371\ 2645 \\ 11.52689\ 5380 & -27.38119\ 0760 & 16.85429\ 5380 \\ 11.52737\ 8125 & -27.38225\ 6250 & 16.85487\ 8125 \\ 11.52786\ 0880 & -27.38332\ 1760 & 16.85546\ 0880 \\ 11.52834\ 3645 & -27.38438\ 7290 & 16.85604\ 3645 \\ 11.52882\ 6420 & -27.38545\ 2840 & 16.85662\ 6420 \\ 11.52930\ 9205 & -27.38651\ 8410 & 16.85720\ 9205 \end{Vmatrix}.$$

It may be noted that since the multiplication is associative, we may start from either left or right to evaluate the product of a series of factors; however, the size of the matrices involved may in the two cases be considerably different. Thus in Example 4 if we start from the left, we continually deal with matrices of 9×3 elements; whereas if we start from the right, we encounter only 3×3 element matrices.

NOTE ON THE GENERAL LORENTZ TRANSFORMATION

By N. Rosen [1]

The purpose of this paper is to determine the pair of quaternion multipliers corresponding to a general rotation of a Cartesian coordinate system in four dimensions, i.e. to the general Lorentz transformation.

It has been shown [2] that a four-dimensional rotation can be represented quite simply by means of quaternion multiplication. If $q = ix_1 + jx_2 + kx_3 + x_4$, is the quaternion with components representing the rectangular coordinates of a point in four dimensions, then the quaternion:

$$q' = p \cdot q \cdot \pi \tag{1}$$

represents a linear transformation of q, where $p = ia + jb + kc + d$ and $\pi = i\alpha + j\beta + k\gamma + \delta$ are quaternions operating on q. The multiplication in (1) is carried out according to the usual rules of quaternions.

However, p and π together involve eight parameters, and as for a general four-dimensional rotation six parameters are required, we impose two restrictions, namely:

$$\left.\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 1, \\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= 1. \end{aligned}\right\} \tag{2}$$

From this it follows that, if $q' = ix_1' + jx_2' + kx_3' + x_4'$, then:

$$x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

which, combined with the fact that for a transformation carried

[1] With indebtedness to Prof. M. S. Vallarta, by whom the problem was suggested, for kind advice and assistance.

[2] For example, c.f., F. Klein's, "Elementarmathematik vom höheren Standpunkte aus," p. 73 et seq (Berlin, 1924).

out in this way the determinant can be shown to be positive, insures a true four-dimensional rotation.

Suppose a transformation, representing a rotation, given by the equations:

$$x_i' = \sum_{j=1}^{4} a_{ij}x_j, \quad (i = 1, 2, 3, 4). \tag{3}$$

This transformation can be represented by

$$x' = Ax$$

where A is the matrix having a_{ij} for the element in the i'th row and j'th column.

To determine the components of p and π corresponding to the matrix A, we carry out the multiplication in (1) and equate the corresponding components on each side of the equation. This gives a system of linear equations of the form (3). From this we can form a matrix, Q. By equating the corresponding elements of the matrices Q and A, there are obtained 16 linear equations with the same number of unknowns, namely the 16 different products of pairs of parameters. These equations can be easily solved by adding or subtracting appropriate pairs. Then having obtained expressions for the products, by suitable division, one can obtain ratios between the components of each quaternion, and taking equations (2) into account, the quaternions are determined. If the matrix A represents a rotation, there are sufficient identities satisfied by its elements to insure consistency of all the equations.

Consider an ordinary space rotation of the coordinates about the X-axis. Taking that direction as positive which corresponds to the rotation of a right-hand screw advancing towards the positive end of the axis. we have, from elementary analytical geometry, for the transformation matrix:

$$A_r = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

where φ is the angle of rotation.

Suppose the axis of rotation M has direction-cosines $\alpha_1, \alpha_2, \alpha_3$. It is most convenient to use vector notation at this point. Take a set of coordinates with X' along M and Z' perpendicular to M and X. Let i, j, k and i', j', k' be the unit vectors in the respective systems. Then we have:

$$i' = i\alpha_1 + j\alpha_2 + k\alpha_3,$$

$$k' = \frac{i \times i'}{\sqrt{1 - \alpha_1^2}} = \frac{k\alpha_2 - j\alpha_3}{\sqrt{1 - \alpha_1^2}},$$

$$j' = k' \times i' = \frac{-i(\alpha_2^2 + \alpha_3^2) + j\alpha_1\alpha_2 + k\alpha_3\alpha_1}{\sqrt{1 - \alpha_1^2}}.$$

If R is the coordinate vector $ix + jy + kz$, we have:

$$x' = i' \cdot R = \alpha_1 x + \alpha_2 y + \alpha_3 z,$$

$$y' = j' \cdot R = -\frac{(\alpha_2^2 + \alpha_3^2)}{\sqrt{1 - \alpha_1^2}} x + \frac{\alpha_1\alpha_2}{\sqrt{1 - \alpha_1^2}} y + \frac{\alpha_1\alpha_3}{\sqrt{1 - \alpha_1^2}} z,$$

$$z' = k' \cdot R = 0 - \frac{\alpha_3}{\sqrt{1 - \alpha_1^2}} y + \frac{\alpha_2}{\sqrt{1 - \alpha_1^2}} z,$$

for which the corresponding matrix is:

$$B = \left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ \dfrac{-(\alpha_2^2 + \alpha_3^2)}{\sqrt{1 - \alpha_1^2}} & \dfrac{\alpha_1\alpha_2}{\sqrt{1 - \alpha_1^2}} & \dfrac{\alpha_1\alpha_3}{\sqrt{1 - \alpha_1^2}} & 0 \\ 0 & \dfrac{-\alpha_3}{\sqrt{1 - \alpha_1^2}} & \dfrac{\alpha_2}{\sqrt{1 - \alpha_1^2}} & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|.$$

Now we apply first the matrix B thus obtaining a system with its X-axis along the axis of rotation, and then the matrix A_r thus carrying out the desired rotation. Then we apply a matrix $\tilde{B}$ obtained by transposing B (i.e. interchanging rows and columns). This restores the system to its original position relative to the axis of rotation. Thus the general rotation has been effected. The matrix R for this is obtained by matrix multiplication:

$$R = \tilde{B} A_r B.$$

Carrying out the multiplication, the matrix for a general three-dimensional space rotation is found to be:

$$R = \begin{Vmatrix} 1 - 2(1-\alpha_1^2)\sin^2\frac{\varphi}{2} & 2\alpha_1\alpha_2\sin^2\frac{\varphi}{2} + 2\alpha_3\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & 2\alpha_3\alpha_1\sin^2\frac{\varphi}{2} - 2\alpha_2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & 0 \\ 2\alpha_1\alpha_2\sin^2\frac{\varphi}{2} - 2\alpha_3\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & 1 - 2(1-\alpha_2^2)\sin^2\frac{\varphi}{2} & 2\alpha_2\alpha_3\sin^2\frac{\varphi}{2} + 2\alpha_1\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & 0 \\ 2\alpha_3\alpha_1\sin^2\frac{\varphi}{2} + 2\alpha_2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & 2\alpha_2\alpha_3\sin^2\frac{\varphi}{2} - 2\alpha_1\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & 1 - 2(1-\alpha_3^2)\sin^2\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Determining the quaternions corresponding to this as described above:[3]

$$p_r = \cos\frac{\varphi}{2} - (i\alpha_1 + j\alpha_2 + k\alpha_3)\sin\frac{\varphi}{2},$$

$$\pi_r = \cos\frac{\varphi}{2} + (i\alpha_1 + j\alpha_2 + k\alpha_3)\sin\frac{\varphi}{2},$$

and it is seen that $p_r = \bar{\pi}_r$. These quaternions have already been found by Klein and Sommerfeld, using a different method.

For a system of coordinates S' initially coinciding with another system S and moving along the X-axis of the latter with a velocity v, we have the following transformation-matrix, which can be shown to represent a rotation in four dimensions about the YZ-plane.[4] Letting $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = \sqrt{-1}\,ct$ where

[3] L. c., p. 78. Also c.f., Klein and Sommerfeld, "Theorie des Kreisels," pp. 16–19 (Leipzig, 1914).

[4] H. Minkowski, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern," p. 10 (Leipzig, 1910).

t is the time coordinate, and c the velocity of light in empty space,

$$\cos\psi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \text{and} \quad \sin\psi = \frac{\sqrt{-1}\,\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}:$$

$$A_e = \begin{Vmatrix} \cos\psi & 0 & 0 & \sin\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\psi & 0 & 0 & \cos\psi \end{Vmatrix}.$$

If the velocity v has direction cosines λ_1, λ_2, λ_3, by going through the same process as above, we get the general Lorentz transformation:[5]

$$L = \begin{Vmatrix} 1-2\lambda_1^2\sin^2\frac{\psi}{2} & -2\lambda_1\lambda_2\sin^2\frac{\psi}{2} & 2\lambda_3\lambda_1\sin^2\frac{\psi}{2} & \lambda_1\sin\psi \\ -2\lambda_1\lambda_2\sin^2\frac{\psi}{2} & 1-2\lambda_2^2\sin^2\frac{\psi}{2} & 2\lambda_2\lambda_3\sin^2\frac{\psi}{2} & \lambda_2\sin\psi \\ -2\lambda_3\lambda_1\sin^2\frac{\psi}{2} & -2\lambda_2\lambda_3\sin^2\frac{\psi}{2} & 1-2\lambda_3^2\sin^2\frac{\psi}{2} & \lambda_3\sin\psi \\ -\lambda_1\sin\psi & -\lambda_2\sin\psi & -\lambda_3\sin\psi & \cos\psi \end{Vmatrix}.$$

The quaternions for this are found to be:

$$p_e = \pi_e = \cos\frac{\psi}{2} + (i\lambda_1 + j\lambda_2 + k\lambda_3)\sin\frac{\psi}{2}.$$

If we perform a rotation, we have:

$$q' = p_r \cdot q \cdot \pi_r.$$

If we then carry out a Lorentz transformation, we get

$$\begin{aligned} q'' &= p_e \cdot q' \cdot \pi_e \\ &= p_e \cdot (p_r \cdot q \cdot \pi_r) \cdot \pi_e \\ &= (p_e \cdot p_r) \cdot q \cdot (\pi_r \cdot \pi_e) \\ &= P_g \cdot q \cdot \Pi_g, \end{aligned}$$

[5] Cf. F. Möglich, "Zur Quantentheorie des rotierenden Elektrons," *ZS. für Phys. 48*, p. 852, 1928.

where $P_\varphi = p_\epsilon \cdot p_r$, $\Pi_\varphi = \pi_r \cdot \pi_\epsilon$, since quaternion multiplication is associative. Carrying out the multiplication:

$$P_\varphi = i\left\{\lambda_1' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} - \alpha_1 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_2\lambda_3' - \alpha_3\lambda_2') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ j\left\{\lambda_2' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} - \alpha_2 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_3\lambda_1' - \alpha_1\lambda_3') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ k\left\{\lambda_3' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} - \alpha_3 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_1\lambda_2' - \alpha_2\lambda_1') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ \cos\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_1\lambda_1' + \alpha_2\lambda_2' + \alpha_3\lambda_3') \sin\frac{\varphi}{2}\sin\frac{\psi}{2},$$

$$\Pi_\varphi = i\left\{\lambda_1' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} + \alpha_1 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_2\lambda_3' - \alpha_3\lambda_2') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ j\left\{\lambda_2' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} + \alpha_2 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_3\lambda_1' - \alpha_1\lambda_3') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ k\left\{\lambda_3' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} + \alpha_3 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_1\lambda_2' - \alpha_2\lambda_1') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ \cos\frac{\varphi}{2}\cos\frac{\psi}{2} - (\alpha_1\lambda_1' + \alpha_2\lambda_2' + \alpha_3\lambda_3') \sin\frac{\varphi}{2}\sin\frac{\psi}{2}.$$

The primed λ's are used because they refer to the system after rotation. By expressing them in terms of the λ's of the system before rotation by means of the matrix R, we obtain:

$$P_\varphi = i\left\{\lambda_1 \cos\frac{\varphi}{2}\sin\frac{\psi}{2} - \alpha_1 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_3\lambda_2 - \alpha_2\lambda_3) \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ j\left\{\lambda_2' \cos\frac{\varphi}{2}\sin\frac{\psi}{2} - \alpha_2 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_1\lambda_3 - \alpha_3\lambda_1) \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ k\left\{\lambda_3 \cos\frac{\varphi}{2}\sin\frac{\psi}{2} - \alpha_3 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_2\lambda_1 - \alpha_1\lambda_2) \sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ \cos\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3) \sin\frac{\varphi}{2}\sin\frac{\psi}{2},$$

$$\Pi_\varphi = i\left\{\lambda_1 \cos\frac{\varphi}{2}\sin\frac{\psi}{2} + \alpha_1 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_3\lambda_2 - \alpha_2\lambda_3)\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ j\left\{\lambda_2 \cos\frac{\varphi}{2}\sin\frac{\psi}{2} + \alpha_2 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_1\lambda_3 - \alpha_3\lambda_1)\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ k\left\{\lambda_3 \cos\frac{\varphi}{2}\sin\frac{\psi}{2} + \alpha_3 \sin\frac{\varphi}{2}\cos\frac{\psi}{2} + (\alpha_2\lambda_1 - \alpha_1\lambda_2)\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\right\}$$
$$+ \cos\frac{\varphi}{2}\cos\frac{\psi}{2} - (\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3)\sin\frac{\varphi}{2}\sin\frac{\psi}{2}.$$

These contain eight parameters, but since there are two identical relations:

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1,$$

only six are independent. Hence these quaternions give the most general four-dimensional rotation.

Conversely, for any four-dimensional rotation matrix one can find the corresponding quaternions. By equating their components to those of P_φ and Π_φ above, it is not difficult to determine φ, ψ, the α's, and the λ's.

AN ANALYSIS OF ROTATIONS IN EUCLIDEAN FOUR-SPACE BY SEDENIONS

By Frank L. Hitchcock

1. Cayley's decomposition. Cayley showed [1] that an orthogonal matrix of the fourth order having determinant $+1$ can be written as the product of two matrices of the form

$$\begin{Vmatrix} \delta & -\alpha & -\beta & -\gamma \\ \alpha & \delta & -\gamma & \beta \\ \beta & \gamma & \delta & -\alpha \\ \gamma & -\beta & \alpha & \delta \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} \delta' & -\alpha' & -\beta' & -\gamma' \\ \alpha' & \delta' & \gamma' & -\beta' \\ \beta' & -\gamma' & \delta' & \alpha' \\ \gamma' & \beta' & -\alpha' & \delta' \end{Vmatrix} \tag{1}$$

wherein

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \alpha'^2 + \beta'^2 + \gamma'^2 + \delta'^2 = 1. \tag{2}$$

He pointed out that the factorization is equivalent to the linear transformation

$$W + iX + jY + kZ = (\delta + i\alpha + j\beta + k\gamma)(w + ix + jy + kz)(\delta' + i\alpha' + j\beta' + k\gamma') \tag{3}$$

if i, j, k obey Hamilton's rules for multiplication. The problem of finding $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ as explicit functions of the elements a_{rs} of a given orthogonal matrix has recently been solved by Mr. N. Rosen.[2] The object of the present paper is to show how the decomposition leads naturally to a sedenion algebra represented by matrices.

2. Matrix algebra. For brevity let the matrix having $+1$ for element in the rth row and sth column and all other elements zero be denoted by e_{rs}. Call these **unit matrices.** The law of multi-

[1] *Recherches Ultérieurs sur les Déterminants Gauches.* Collected Papers, vol. II, p. 214. Crelle's *Jour.*, vol. 50, (1855), pp. 299–313. See Art. 6 below.

[2] This Journal, this Volume and No., p. 181.

plication for unit matrices is

$$e_{rs}e_{st} = e_{rt}, \qquad e_{rs}e_{uv} = 0 \quad \text{if} \quad s \neq u. \tag{4}$$

Any matrix can be written as a linear aggregate of unit matrices, $A = \sum a_{rs}e_{rs}$, $B = \sum b_{pq}e_{pq}$, etc. The product AB can be found by the distributive law of multiplication.

3. Special matrices. The two matrices (1) can be written

$$\begin{aligned}\delta(e_{11} + e_{22} + e_{33} + e_{44}) + \alpha(e_{21} - e_{12} + e_{43} - e_{34}) \\ + \beta(e_{31} - e_{13} + e_{24} - e_{42}) + \gamma(e_{41} - e_{14} + e_{32} - e_{23}) \\ = \delta E + \alpha I + \beta J + \gamma K,\end{aligned} \tag{5}$$

say; and also

$$\begin{aligned}\delta'(e_{11} + e_{22} + e_{33} + e_{44}) + \alpha'(e_{21} - e_{12} - e_{43} + e_{34}) \\ + \beta'(e_{31} - e_{13} - e_{24} + e_{42}) - \gamma'(-e_{41} + e_{14} + e_{32} - e_{23}) \\ = \delta'E + \alpha'I' + \beta'J' - \gamma'K'.\end{aligned} \tag{6}$$

The special matrices I, J, K, E obey the same rules of multiplication as $i, j, k, 1$; and the same holds for I', J', K', E.[3] Because the factors in (3) are associative, I, J, K must be commutative with I', J', K', respectively, as may be verified directly; that is, the matrices (1) are commutative.

The products of I, J, K, with I', J', K' are shown in the following multiplication table:

	I'	J'	K'
I	$-e_{11}-e_{22}+e_{33}+e_{44}$	$e_{14}-e_{23}-e_{32}+e_{41}$	$e_{13}+e_{24}+e_{31}+e_{42}$
J	$-e_{14}-e_{23}-e_{32}-e_{41}$	$-e_{11}+e_{22}-e_{33}+e_{44}$	$-e_{12}-e_{21}+e_{34}+e_{43}$
K	$e_{13}-e_{24}+e_{31}-e_{42}$	$-e_{12}-e_{21}-e_{34}-e_{43}$	$e_{11}-e_{22}-e_{33}+e_{44}$

(7)

It can be seen by inspection that the 16 matrices E, I, J, K, I', J', K', II', IJ', $\cdots$, KK' are linearly independent, for if the e_{rs} be

[3] It is of great geometric interest that I, J, K are self-complementary, while I', J', K', are anti-self-complementary. These facts have been made the basis of a highly elegant system of vector analysis in Euclidean 4-space by Rev. Father F. W. Sohon, S.J.;—and also of an analysis of finite rotation *groups* in this space, (This JOURNAL, this Vol. and No., p. 194). I take this opportunity of expressing my appreciation of valuable suggestions from him, and from his hitherto unpublished notes.

thought of as basis of a Euclidean 16-space, these matrices are orthogonal vectors. They may be taken as elements of a *sedenion algebra*, and an arbitrary matrix A of order four may be written as a linear aggregate,

$$A = c_E E + c_1 I + c_2 J + c_3 K + c_1' I' + c_2' J' + c_3' K' + c_{11} II' + c_{12} IJ' + \cdots + c_{33} KK'. \quad (8)$$

where $c_E, c_1, c_2, c_3, c_1', c_2', c_3', c_{11}, \cdots, c_{33}$ are arbitrary coefficients. If A is *given*, any c can be written at once by starting with the matrix it multiplies, changing e_{rs} into a_{rs}, and dividing by 4. Example: $c_{12} = \frac{1}{4}(a_{14} - a_{23} - a_{32} + a_{41})$.

4. Consequences of Cayley's decomposition. If A corresponds to a rotation we know that (8) factors into (5) and (6); and hence that the matrix

$$\begin{Vmatrix} c_E & c_1' & c_2' & c_3' \\ c_1 & c_{11} & c_{12} & c_{13} \\ c_2 & c_{21} & c_{22} & c_{23} \\ c_3 & c_{31} & c_{32} & c_{33} \end{Vmatrix} \quad (9)$$

is of rank 1.

I shall now show that if (9) is of rank 1, and if the sum of the squares of the 16 coefficients $c_E \cdots c_{33}$ is 1, the matrix A is orthogonal.

Proof. A sufficient condition for orthogonality is $AA' = E$ where A' is the conjugate or transposed matrix of A. The terms in I, J, K, I', J', K' are antisymmetrical, the others symmetrical, hence

$$A' = c_E E - c_1 I - c_2 J - c_3 K - c_1' I' - c_2' J' - c_3' K' + c_{11} II' + \cdots + c_{33} KK'. \quad (10)$$

If we now multiply out AA' it is not difficult (by aid of the multiplication table for sedenions given below) to pick out the coefficients of the sixteen terms in the product and find that

1°. The coefficient of E is the sum of the squares of $c_E, \cdots, c_{33}$.

2°. The other coefficients either vanish identically or consist of sums or differences of second order minors of (9). They accordingly vanish by hypothesis. The proposition is therefore established.

As an interesting corollary we may note that, under the same hypotheses

$$(c_{11}II' + c_{12}IJ' + \cdots + c_{33}KK')^2 = (c_{11}^2 + \cdots + c_{33}^2)E. \quad (11)$$

Furthermore, these hypotheses insure that the determinant of A shall be $+1$. For we can at once factor into two matrices of form (1) by noting that $\alpha, \beta, \gamma, \delta$ are proportional to the elements of any column in (9) not all zero; and $\alpha', \beta', \gamma', \delta'$, are proportional to the elements of any row in (9) not all zero. By using $AA' = E$ it can then be shown that the determinant of A is $+1$.

5. Group properties of the sedenion expansion (8). If the right member of (8) be spoken of as a vector in sedenion space, and if a rotation be performed in the original four-space, there are several components of (8) which may be said to determine *subspaces* in the sedenion space. The term $c_E E$ is invariant. The three terms $c_1I + c_2J + c_3K$ lie in a subspace in the sense that any "vector" of this form becomes another of the same form. For let the rotation be determined by two matrices R and R' of types (5) and (6) respectively. Let $c_1I + c_2J + c_3K = V$, which becomes $(RR')V(RR')^{-1}$ when the four-space is transformed by RR', and simplifies to RVR^{-1} because R' is commutative with V and R. Formally R is a quaternion and V is a vector, hence RVR^{-1} is V rotated through an angle in the "three-space" of I, J, K,[4] a fact easily verified by multiplying out. The "norm" $c_1^2 + c_2^2 + c_3^2$ is invariant.

A similar demonstration applies to the terms $c_1'I' + c_2'J' + c_3'K'$. The other nine terms $c_{11}II' \cdots c_{33}KK'$ certainly lie in a subspace because they comprise all the symmetrical terms except the invariant $c_E E$. What has so far been said in Art. 5 is true of an arbitrary matrix A of order 4. If we now restrict A to denote a rotation, so that (9) is of rank unity, the nine terms $c_{11}II' + \cdots + c_{33}KK'$ factor into two matrices which we may call V_1 and V_1', the first being a linear function of I, J, K,

[4] Hamilton, *Elements*, Art. 297.

the second a linear function of I', J', K'. If, then, A is transformed into $RR'AR'^{-1}R^{-1}$ the nine terms are transformed into $RV_1R^{-1}R'V_1'R'^{-1}$ which is of the same type as V_1V_1'. The norm of V_1, as well as of V_1', is invariant.

6. Validity of Cayley's decomposition.[5] Any second order minor from (9) is easily seen to vanish if A is orthogonal of determinant $+1$ by expressing the c's in terms of the a's as in Art. 3, using the theorem that any minor of A equals its algebraic complement. Hence (9) is of rank 1 and the decomposition is possible.

Ex. 1. $16(c_1c_{21} - c_2c_{11})$

$$= (a_{21} - a_{12} + a_{43} - a_{34})(-a_{14} - a_{23} - a_{32} - a_{41})$$
$$- (a_{31} - a_{13} + a_{24} - a_{42})(-a_{11} - a_{22} + a_{33} + a_{44})$$
$$= \Sigma a_{r2}a_{r4} + \Sigma a_{1r}a_{3r} - \Sigma a_{r1}a_{r3} - \Sigma a_{2r}a_{4r} + 1223 - 3144$$
$$+ 1241 - 2433 + 2231 - 4314 + 1124 - 3243 = 0,$$

where $pqrs$ means $a_{pq}a_{rs} - a_{ps}a_{rq}$.

Ex. 2. $16(c_2c_{31} - c_3c_{21})$

$$= (a_{31} - a_{13} + a_{24} - a_{42})(a_{13} - a_{24} + a_{31} - a_{42})$$
$$- (a_{41} - a_{14} + a_{32} - a_{23})(-a_{14} - a_{23} - a_{32} - a_{41})$$
$$= (a_{31}^2 + a_{32}^2) + (a_{41}^2 + a_{42}^2) - (a_{23}^2 + a_{13}^2)$$
$$- (a_{24}^2 + a_{14}^2) + 2[3241 - 1423]$$
$$= (a_{31}^2 + a_{32}^2) + (a_{41}^2 + a_{42}^2) + (a_{33}^2 + a_{43}^2 - 1)$$
$$+ (a_{34}^2 + a_{44}^2 - 1) = 0.$$

7. The multiplication table for sedenions. The commutative property of the Cayley factors, together with Hamilton's rules, make it a simple matter to write out the multiplication table for sedenions. Example: $IJ'KI' = IKJ'I' = (-J)(-K') = JK'$. The table follows. The first row and first column serve the double purpose of denoting multiplicand and multiplier, respectively, and also their products by and into E. The compound elements II' etc. are in heavy type as aid to the eye.

[5] A flaw in Cayley's reasoning, investigated by Netto, *Act. math.*, Bd. 9 (1887), p. 295, is that Cayley's 8-parameter form, though always valid, is derived from 6- and 7-parameter forms which are not. Cayley's statement, *Col. Works*, II, p. 108, *Phil. Mag.*, vol. VI (1853), circa, p. 326, that the 6-parameter form is always valid neglects possible vanishing of divisor, which, to be sure, is a sum of squares. Netto proved the 7-parameter forms may fail when $|A + E| = 0$.

MULTIPLICATION TABLE FOR SEDENIONS

E	I	J	K	I'	J'	K'	II'	IJ'	IK'	JI'	JJ'	JK'	KI'	KJ'	KK'
I	$-E$	K	$-J$	II'	IJ'	IK'	$-I'$	$-J'$	$-K'$	KI'	KJ'	KK'	$-JI'$	$-JJ'$	$-JK'$
J	$-K$	$-E$	I	JI'	JJ'	JK'	$-KI'$	$-KJ'$	$-KK'$	$-I'$	$-J'$	$-K'$	II'	IJ'	IK'
K	J	$-I$	$-E$	KI'	KJ'	KK'	JI'	JJ'	JK'	$-II'$	$-IJ'$	$-IK'$	$-I'$	$-J'$	$-K'$
I'	II'	JI'	KI'	$-E$	K'	$-J'$	$-I$	IK'	$-IJ'$	$-J$	JK'	$-JJ'$	$-K$	KK'	$-KJ'$
J'	IJ'	JJ'	KJ'	$-K'$	$-E$	I'	$-IK'$	$-I$	II'	$-JK'$	$-J$	JI'	$-KK'$	$-K$	KI'
K'	IK'	JK'	KK'	J'	$-I'$	$-E$	IJ'	$-II'$	$-I$	JJ'	$-JI'$	$-J$	KJ'	$-KI'$	$-K$
II'	$-I'$	KI'	$-JI'$	$-I$	IK'	$-IJ'$	E	$-K'$	J'	$-K$	KK'	$-KJ'$	J	$-JK'$	JJ'
IJ'	$-J'$	KJ'	$-JJ'$	$-IK'$	$-I$	II'	K'	E	$-I'$	$-KK'$	$-K$	KI'	JK'	J	$-JI'$
IK'	$-K'$	KK'	$-JK'$	IJ'	$-II'$	$-I$	$-J'$	I'	E	KJ'	$-KI'$	$-K$	$-JJ'$	JI'	J
JI'	$-KI'$	$-I'$	II'	$-J$	JK'	$-JJ'$	K	$-KK'$	KJ'	E	$-K'$	J'	$-I$	IK'	$-IJ'$
JJ'	$-KJ'$	$-J'$	IJ'	$-JK'$	$-J$	JI'	KK'	K	$-KI'$	K'	E	$-I'$	$-IK'$	$-I$	II'
JK'	$-KK'$	$-K'$	IK'	JJ'	$-JI'$	$-J$	$-KJ'$	KI'	K	$-J'$	I'	E	IJ'	$-II'$	$-I$
KI'	JI'	$-II'$	$-I'$	$-K$	KK'	$-KJ'$	$-J$	JK'	$-JJ'$	I	$-IK'$	IJ'	E	$-K'$	J'
KJ'	JJ'	$-IJ'$	$-J'$	$-KK'$	$-K$	KI'	$-JK'$	$-J$	JI'	IK'	I	$-II'$	K'	E	$-I'$
KK'	JK'	$-IK'$	$-K'$	KJ'	$-KI'$	$-K$	JJ'	$-JI'$	$-J$	$-IJ'$	II'	I	$-J'$	I'	E

ROTATION AND PERVERSION GROUPS IN EUCLIDEAN SPACE OF FOUR DIMENSIONS

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I. QUATERNION VERSOR GROUPS

1. Rotation Groups in Space of Three Dimensions. The rotation groups in ordinary space of three dimensions fall into three divisions. In the first division only one axis of rotation is permitted, and different rotation groups are formed by allowing

different amounts of rotation about this axis. These groups are all cyclic groups. In the rotation groups of the second division there are lateral axes perpendicular to one principal axis. The only rotations permitted about the lateral axes are rotations through 180°, but the rotation about the principal axis may be any aliquot part of 360°. In obtaining different groups in the second division, the angle of rotation about the principal axis is the only thing varied. It is obvious that in the first and second divisions the number of rotation groups will be unlimited. The third division consists in those groups having more than one axis about which a rotation different from 180° is allowed. It can be proved that there are only three rotation groups in the third division, and the proof of this theorem is here assumed. The rotation groups of the first division are exemplified by the third order forms of the hemi-pyramids in crystallography.* The rotation groups of the second division are found in the forms of trapezohedral symmetry, except that where the principal axis allows only a rotation of 180° only and thus becomes indistinguishable from this standpoint from a lateral axis, the figure obtained is called the rhombic bisphenoid. Only two of the three groups of the third division occur in crystallography. The tetartoid has four axes about which rotations of 120° are allowed, and three axes about which rotations of 180° alone are permitted. The gyroid adds to the tetartoid six more axes for rotations of 180°, and allows rotations of 90° about the three original 180° axes. The third and last rotation group of this division occurs in the regular dodecahedron where it is combined with perversions.

2. Quaternion Versor Groups. Rotations in space of three dimensions may be treated by means of quaternions, the operator $q^{-1}(\ )q$ where q is a quaternion being equivalent to a rotation through twice the angle of the quaternion. If we confine the discussion to quaternion versors, then for every pair of quaternion

* Every different author seems to have different names for the groups of symmetry in space of three dimensions. The names here used are taken from Moses and Parsons, Mineralogy, Crystallography and Blow-Pipe Analysis.

versors $+q$ and $-q$ there corresponds a unique rotation and vice versa, so that except for sign the multiplication table for rotations will also serve for quaternion versors. A complete discussion of quaternion versor groups can therefore be based on the rotation groups in space of three dimensions. There are rotations in space of four dimensions that have the same properties as quaternions. Hence the investigation of quaternion versor groups is at the same time a beginning of the discussion of rotations in space of four dimensions.

3. Quaternion Versor Groups in Four Space. Transformation of a figure into itself will be represented by E. Orthogonal transformations U in space of four dimensions having the property $U^2 = -E$ will be called *Hamilton Units*. In particular the three Hamilton Units that would be used to denote the rotations of the rhombic bisphenoid will be called *Sphenoidals*, and will be represented by I, J, and K. (Cf. infra Art. 12.) Hamilton's multiplication table is used.

$$IJ = K = -JI, \qquad JK = I = -KJ, \qquad KI = J = -IK,$$
$$I^2 = J^2 = K^2 = -E.$$

An orthogonal transformation of the form $xI + yJ + zK + wE$ where $x^2 + y^2 + z^2 + w^2 = 1$ is called a *quaternion versor* in this discussion, for it has all the formal properties of such. The quaternion versor is also written in the form $E \cos q + U \sin q$ where U is a Hamilton Unit. The angle q will always be an aliquot part of 360°, and the quaternion versor groups corresponding to the first division of rotation groups in space of three dimensions will each consist of a single quaternion versor together with its powers.

4. Whorls. In order to obtain a group corresponding to the second division of three dimensional groups, we take the quaternion versor $H = E \cos q + K \sin q$ and the Hamilton Unit I. The rules for multiplying these transformations are as follows:

$$IH^t = I \cos qt - J \sin tq = H^{-t}I.$$

The group will therefore consist of two series of transformation,

one series represented by H^t and the other consisting of Hamilton Units represented by IH^t where t takes on all integral values. Such a group is called a whorl. The versor H is called the *stem* of the whorl, and the Hamilton Units are called *peripherals*. It is important to have a test as to whether a given Hamilton Unit will do as peripheral for a given versor. Now it is quite clear that if we examine the coefficient of E in the product UH and find that it vanishes, the series UH^t will consist entirely of Hamilton Units.

Put

$$(UH^t)^2 = -E,$$

then

$$\begin{aligned} UH^tUH^t &= -E, \\ UH^tU &= -H^{-t}, \\ UH^t &= H^{-t}U, \end{aligned}$$

and the same multiplication table is obtained. Hence we may generalize and formulate the following definitions:

A Hamilton Unit is said to be *peripheral* to another quaternion versor, if the product of the two is another Hamilton Unit, that is, if the coefficient of E in the product UH vanishes.

A group of rotations developed from a quaternion versor and a peripheral (Hamilton Unit) is called a *whorl*.

5. The Tetartoid Umbel. There remain three more quaternion groups to be constructed to correspond to the rotation groups of division three. Quaternion versor groups that are neither cyclic groups nor whorls will be called *umbels*.

Corresponding to the tetartoid we define a group called the *tetartoid umbel* consisting of the sphenoidals I, J, K and four new versors defined as follows:

$$\begin{aligned} A &= \tfrac{1}{2}(E - I - J - K), & C &= \tfrac{1}{2}(E + I - J + K), \\ B &= \tfrac{1}{2}(E + I + J - K), & D &= \tfrac{1}{2}(E - I + J + K). \end{aligned}$$

It will be convenient to call these new versors *tetartoidals*. The sphenoidals are of the fourth order, $I^4 = J^4 = K^4 = E$ and the tetartoidals are of the sixth order, $A^6 = B^6 = C^6 = D^6 = E$.

The tetartoid umbel is a rotation group containing 24 members, sc. A, B, C, D, I, J, K and their powers, and a multiplication table for it is contained in the multiplication table for the following umbel in which the tetartoid umbel plays the part of a subgroup.

6. The Gyroid Umbel. Corresponding to the gyroid we construct a *gyroid umbel*. Two new sorts of transformation are required. The *cardinals* are defined

$$X = \frac{E+I}{\sqrt{2}}, \qquad Y = \frac{E+J}{\sqrt{2}}, \qquad Z = \frac{E+K}{\sqrt{2}},$$

and peripheral to these we have six new Hamilton Units to be called *gyroidals*:

$$F = \frac{J-K}{\sqrt{2}}, \qquad N = \frac{K-I}{\sqrt{2}}, \qquad P = \frac{I-J}{\sqrt{2}},$$

$$G = \frac{J+K}{\sqrt{2}}, \qquad M = \frac{K+I}{\sqrt{2}}, \qquad Q = \frac{I+J}{\sqrt{2}}.$$

A multiplication table has been constructed for the gyroid umbel.

7. The Icosahedral Umbel. The construction of a quaternion versor group to correspond to the rotation group of the regular icosahedron is a much more elaborate affair. The group has 120 transformations and a multiplication table would be too cumbersome. The treatment is therefore a little different from the preceding. One new versor of the tenth order is defined:

$W = \frac{1}{4}\{E(1+\sqrt{5}) + 2I + (1-\sqrt{5})K\}$ abbreviated $\{E\,I\,K\}$.

The 120 transformations are obtained by applying powers of W

$$W^2 = \{I\,-K\,-E\}, \qquad W^3 = \{I\,-K\,E\}, \qquad W^4 = \{-E\,I\,K\},$$

as postfactors to the 24 transformations of the tetartoid umbel. For algebraic work the following rules are of service:

$$\{E\ E\ E\} = 2\{0\ E\ 0\}, \qquad \{E\ E\ -E\} = 2\{E\ 0\ 0\},$$
$$\{E\ -E\ E\} = 0, \qquad \{-E\ E\ E\} = 2\{0\ 0\ E\},$$
$$\{0\ E\ 0\} = \tfrac{1}{2}E,$$
$$\{E\ 0\ 0\}^2 = \tfrac{1}{2}\{E\ E\ 0\}, \qquad \{E\ 0\ 0\}\{0\ 0\ E\} = \tfrac{1}{2}\{0\ -E\ 0\},$$
$$\{0\ 0\ E\}^2 = \tfrac{1}{2}\{0\ E\ E\}.$$

Tables are given to enable the transformations to be expressed either in terms of E, I, J and K, or in terms of the transformations of the tetartoid umbel with the decimal versor powers either as prefactors or as postfactors.

8. Types and Species. The transformation of a transformation by a transformation will be called a conversion. Thus the operator $R^{-1}(\ )R$ will be called a *converter*, and the result obtained $R^{-1}SR$ will be called the *conversion* of S by R. If R is allowed to take on all possible values for which a reciprocal exists, then the values obtained for $R^{-1}SR$ constitute the *species* of S. Transformations of the same species are said to be *similar*. If R is restricted to a transformation of the group under discussion, then the conversions obtained constitute the *type* of S in that particular group. A theorem proved for a particular transformation generalizes at once by conversion to all the transformations of that type, but not necessarily holds for all transformations of that species. In the tetartoid umbel the tetartoidals and their reciprocals belong to different types, and therefore in the tetrahedron one end of the three fold axis emerges in a vertex while the other emerges in a face. In the gyroid umbel the sphenoidals and the gyroidals both belong to the same species for they are both Hamilton Units. But they belong to different types, and as a consequence some of the planes of symmetry of the cube emerge in edges, and some in the middle of faces. When the tetartoidals are taken as part of the gyroid umbel they are found to be of the same type as their reciprocals and in the cube opposite ends of the threefold axes emerge alike. In the icosahedral umbel very fortunately the types and species are co-extensive. Tables for conversions have been constructed, and the types may be determined immediately by an inspection of the tables. For whorls the following formulas are easily derived, from the last two of which we conclude that alternate peripherals are of different types, as $2n$ will always be an even number.

$$H^{-m}H^nH^m = H^n,$$
$$(IH^m)^{-1}H^n(IH^m) = H^{-n},$$
$$H^{-m}IH^nH^m = IH^{2m+n},$$
$$(IH^m)^{-1}IH^n(IH^m) = IH^{2m-n}.$$

The stem of a whorl may, without loss of generality, be taken as of even order, because the square of a peripheral, $-E$, may be multiplied into the stem where the latter is of odd order, and the product will be a quaternion versor of twice the order of the original stem.

II. Rotations and Perversions

9. Versors and Perverters. A *versor* is defined to be an orthogonal transformation with determinant $+1$. A *perverter* is defined to be an orthogonal transformation with determinant -1. The result of applying a versor to a figure is called a *rotation*. The result of applying a perverter to a figure is called a *perversion*. The locus of points left unchanged by a transformation is called the *stator* of that transformation. Thus in space of four dimensions the stator of a versor will in general be a point, but sometimes a plane, and the stator of a perverter will in general be a line but sometimes a planoid or spread of three dimensions. A versor whose stator is a plane is called a *simple versor*. A perverter whose stator is a planoid is called a *simple perverter*. A plane that is so transformed that its direction or aspect remains unchanged, but in which every point undergoes rotation in the plane itself, is called a *rotor plane*. A simple versor has one rotor plane normal to its one stator plane. Other versors, except quaternion versors, may be considered as having a pair of normal rotor planes. The perverter in general will be found to have a rotor plane perpendicular to its stator when the latter is a line. The line perpendicular to the stator of a simple perverter, and the line perpendicular to the planoid containing rotor and stator in the case of other perverters is called the *axis of perversion*. Lines parallel to the axis of perversion have their directions reversed by the perverter.

10. Elements of a Versor. By the choice of a suitable converter $\Phi^{-1}(\ )\Phi$ any versor can be converted into the form of the following matrix:

$$\begin{Vmatrix} \cos q & \sin q & 0 & 0 \\ -\sin q & \cos q & 0 & 0 \\ 0 & 0 & \cos q' & \sin q' \\ 0 & 0 & -\sin q' & \cos q' \end{Vmatrix}.$$

It is obvious that this matrix can be written as the product of two factors

$$\begin{Vmatrix} \cos q & \sin q & 0 & 0 \\ -\sin q & \cos q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos q' & \sin q' \\ 0 & 0 & -\sin q' & \cos q' \end{Vmatrix}.$$

Each factor represents a simple versor. If we apply the converter $\Phi(\ )\Phi^{-1}$ to each factor, the product returns to the original versor which is now expressed as the product of two simple versors. We have then the theorem that in general a versor can be expressed as the product of two simple versors, and except in the case of quaternion versors there is only one way in which the factorization into simple versors can be effected into factors that will be commutative with each other. The two commutative simple versors into which a given versor can be factored are called the *elements* of the versor. The term *compound versor* can be used for any versor other than a simple versor, but will not usually be applied to quaternion versors because the elements of a quaternion versor are indeterminate.

11. Elements of a Perverter. The general case of the perverter can be converted into the matrix

$$\begin{Vmatrix} \cos q & \sin q & 0 & 0 \\ -\sin q & \cos q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix},$$

which can be written as the commutative product of a simple versor and the simple perverter represented by the matrix:

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

We can now define the *elements* of a perverter to be the pair of commutative factors, one of which is a simple versor and the other a simple perverter, into which the given perverter can be factored. A perverter that is not simple is called *compound*.

12. Two Classes of Quaternions. The quaternion versors that we have been considering and which we have written in the form $wE + xI + yJ + zK$ will be written in matrix form

$$H = \begin{Vmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{Vmatrix},$$

and the special forms of each transformation considered can be gotten by substituting for w, x, y, and z their respective values in the particular case in question. We next introduce a fundamental perverter which may be defined by the matrix

$$\Pi = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

If this transformation is used to convert the quaternion versor $wE + xI + yJ + zK$ we obtain another quaternion versor of the same species represented by the matrix

$$H' = \begin{Vmatrix} w & -x & -y & z \\ x & w & -z & y \\ y & z & w & x \\ -z & y & -x & w \end{Vmatrix} = \Pi H \Pi,$$

which we shall write $wE' + xI' + yJ' + zK'$. To attach a prime to quaternion versor is to be interpreted in every case to have the effect of applying the conversion II( )II. The unprimed quaternion versors will be called *quaternion versors of the first class*, while the primed versors will be called *quaternion versors of the second class*. If there are no perverters in a group, a distinction of type will exist between the two classes, but if perverters are present the types will include perverters of both classes.

13. The Commutative Property. It can be proved that if a product be formed using two quaternion versors of different classes, this product will always be commutative, and the proof of this is here supposed. Quaternions of the same class are not commutative unless they have the same Hamilton Unit, and therefore unless one is a power (not necessarily integral) of the other.

14. Quaternion Factors of a Versor. Besides the factorization of a versor into the product of two commutative simple versors just given, it has been shown by Cayley (Col. Papers II, p. 214) that every rotation in space of four dimensions can also be factored into the product of two quaternion versors of different classes. Except for the signs of the factors (since two negative signs compensate for each other) the resolution of a versor into quaternion versor factors is unique. There is a simple relation between the angles of the quaternion versor factors and the angles of the simple versor factors of a given versor. The sum of the angles of the quaternion factors will give the angle of one simple versor factor, while the difference between the angles of the quaternion factors gives the angle of the other simple versor factor. In other words if we write

$$\Omega = (E \cos q_1 + U \sin q_1)\,(E \cos q_2 + V' \sin q_2),$$

then the elements of the versor Ω will have the angles $q = q_1 + q_2$ and $q' = q_1 - q_2$ respectively. It can also be shown that if the Hamilton Units U and V' are not varied, the rotor planes of the

versor Ω will not be changed by any change in the angles q_1 and q_2. For geometrical interpretation the simple versor factors are of fundamental importance, while for algebraic manipulation the quaternion versor factors are more important.

15. Composite Nature of Rotation Groups. If we wish to find the product of two rotations say $(AQ')(PX')$, then since multiplication is associative we omit the parentheses and write $AQ'PX'$. The two factors Q' and P are commutative so that the product can be written $APQ'X'$ and again $(AP)(Q'X')$ on account of the associative property. In other words *the quaternion versors of different classes multiply together separately and independently.* Hence *if the rotations themselves form a group, then the quaternion factors of the first class must form a group among themselves, and the quaternion versors of the second class must likewise form a group.* For in forming all possible products of the rotations we also form all possible products of their respective factors among themselves. This means that in every rotation group, *the quaternion factors of each class must be assembled from one of the quaternion groups that we have discussed.* To know *all* possible rotation groups we must therefore know *all the ways of combining quaternion versors groups of different classes.*

16. Perverter Groups. The product of two versors is always a versor. The product of two perverters is always a versor. The product of a versor and of a perverter in either order is always a perverter. If we have a group containing perverters and drop the perverters from the group, we have dropped nothing that could have been formed by the versors, so that the transformations that remain must form a rotation group. If we know all the rotation groups, the question of finding the perversion groups resolves itself into finding how to add perverters to a group in such a way that no new rotations will be produced. Introducing perverters into a rotation group will be spoken of as perverting the rotation group. The number of perverters and the number of versors will always be equal because if Ψ is a perverter and Ω is a versor, then the product $\Psi\Omega$ will be different

for every different value of Ω and as every possible perverter is obtained from Ψ by the application of some versor as postfactor, and the latter may be recovered by applying Ψ^{-1} as prefactor, then all the perverters of the group must be obtained by applying as postfactors to Ψ the various versors of the group. It is only necessary therefore to insert the first perverter into a rotation group and the rest will be supplied by the multiplication of the various versors. It will usually be found possible to pervert a given rotation group in more than one way.

17. The Perverter Test. Since any perverter applied as converter converts a quaternion of one class into a similar quaternion of the other class, the question of the introduction of perverters does not arise except in these rotation groups that have been formed from two similar (simply isomorphic) quaternion versor groups. Supposing such a group, the square of the perverter should first be examined to make sure that it is a member of the rotation group itself. Let us suppose this condition also to be satisfied. Let Φ and Ω be any two rotations of the group. Form the perverters $\Psi\Phi$ and $\Psi\Omega$ where Ψ is the perverter it is proposed to introduce. The product $\Psi\Omega\Psi\Phi$ is the general expression for any versor produced as a result of the introduction of Ψ. As second test this product must for all values of Ω and Φ be a versor of the group. If the product be multiplied by versors that are already members of the group, then the new product will be a member of the group when and only when the old one was a member of the group. In particular, we multiply it by Φ^{-1} as postfactor, and by Ψ^{-2} as prefactor, and thus obtain the second part of the test. Hence to determine whether Ψ is admissible to a group as perverter, we first test the square to see if Ψ^2 is already a member of the group, and then we apply the conversion test to see if for all values of Ω the conversion $\Psi^{-1}\Omega\Psi$ is still a member of the group. This is not the simplest test, but it is the easiest because we have tables of conversions already constructed.

III. Blending of Quaternion Versors

18. Graphical Relations. The manner in which the properties of the rotation

$$(E \cos q_1 t + U \sin q_1 t)(E \cos q_2 t + V' \sin q_2 t)$$

are controlled by the angles q_1 and q_2 is best by means of a diagram in which the angles q_1 and q_2 are chosen as abscissas and ordinates respectively. Admitting angles greater than 360° divides the plane into a checkerboard of squares of side 360°. The transformation E is represented by the corner of every square, as well as by the center of each square. In the latter case we have the product $(-E)(-E)$. The middle point of each side of each square represents $-E$. The points plotted along the sides of the squares represent quaternion versors. The points plotted along the diagonals of the squares represent simple versors, since for these the sum or difference of the quaternion angles vanish. Parallels to the sides through the middle points divide the square into quarters. If our rotation is located in one quarter and we add 180° to q_1 we move into the adjacent quarter. But we have only changed the sign of the rotation. Hence the transformations in adjacent quarters differ only in sign. From this it follows that all quarters situated diagonally from each other are identical, and each rotation is plotted twice in the same large square. On the other hand ii the angles $q = q_1 + q_2$ and $q' = q_1 - q_2$ are taken as co-ordinates, in other words take the squares formed by the diagonals, whose areas are half the area of the first square, then a rotation only occurs once in such a square. For this reason compound rotations should be described by q and q', not by q_1 and q_2.

19. Conversions. To determine the number of each species we have recourse to conversions. How many ways can the graph be transformed so as to keep the E transformations in the same place? Eight, naturally. The conveters are E, II', I, I', Π, $\Pi\, II'$, $\Pi\, I$, $\Pi\, I'$. But we already know that a transformation is of the same species as its reciprocal, hence we

have only three to consider I, Π, and $\Pi\, I$. A table of the conversions of quaternion versors and of simple versors is given:

$E(\ )E$	H	H'	$\theta = HH'$	$\theta^* = HH'^{-1}$
$II'(\ )II'$	H^{-1}	H'^{-1}	θ^{-1}	θ^{*-1}
$-\ I(\ )I$	H^{-1}	H'	θ^{*-1}	θ^{-1}
$-\ I'(\ )I'$	H	H'^{-1}	θ^*	θ
$\Pi(\ )\Pi$	H'	H	θ	θ^{*-1}
$(\Pi\, II')^{-1}(\ )(\Pi\, II')$	H'^{-1}	H^{-1}	θ^{-1}	θ^*
$(\Pi\, I)^{-1}(\ )(\Pi\, I)$	H'	H^{-1}	θ^{*-1}	θ
$(\Pi\, I')^{-1}(\ )(\Pi\, I')$	H'^{-1}	H	θ^*	θ^{-1}

The I conversion calls for the interchange of the planes of rotation. Two groups that are so related that they are converted into one another this way are said to be *complementary*. A group may be self-complementary.

The Π conversion replaces one simple versor with its reciprocal, but leaves the other unchanged. It interchanges the quaternions. The interchange of quaternions is called reversal, and a group that is not affected by reversal is said to be *reversible*. Simple versors and alternators are reversible as well as various combinations of these.

The $\Pi\, I$ conversion a simultaneous interchange of rotor planes and of quaternions A group that is not affected by this conversion is said to be *quadrantal*. This property is only of importance in connection with the perversion of whorls (Cf. Art. 36).

20. Cyclic Groups. The simultaneous presence of rotations of an object produces symmetry. The following names are suggested:

Isotropic n-fold Symmetry is described by the cyclic group formed from a quaternion versor of order n. In particular if n equals 2, the group describes the presence of a center of symmetry.

A Plane of n-fold Symmetry is described by the presence of a cyclic group formed from the simple versor HH' or HH'^{-1} of the nth order. It is spoken of as having a rotor and a stator plane.

Alternating n-fold Symmetry is described when the generating transformation is the negative of a simple versor of order greater than 2. Such a versor is called an alternator. There is one rotor plane, but no stator plane. Alternators are always of even order. Alternators of oddly even order occur in pairs, one derived from a simple versor of equal order, one derived from the odd simple versor of half the order. Ambiguity may be avoided by adhering to the rule of naming the alternator from its elements, but it seems better to regard such versors as combinations of an odd plane with either a center of symmetry or else a plane of twofold symmetry and name them so.

Perverters. The remarks just made also apply to perverters. The perverters of evenly even order can be named from their order, but the two species of oddly even order must be named from their elements, or else one considered as the combination of an odd plane with a simple perverter, that is to say, a planoid of twofold symmetry, the other as a combination of an odd plane with a line of twofold symmetry.

Quasi-isotropic Symmetry is described by the cyclic group formed from the product of a Hamilton Unit and a quaternion versor of higher order. If the order of the second versor is a multiple of eight, the resulting order will be exactly the same, if an odd multiple of four, one half the same, if oddly even, twice as great, if odd four times as great. Where the final order is an odd multiple of four, the differences that appear are a matter of notation only, so that no ambiguity arises in naming the groups from their order, which is the same thing as naming them from their elements. There are two rotor planes, as in the following case.

Compound n-fold Symmetry. If the cyclic groups already named be cancelled, then if we do not exceed the eighth order, there remains one species of compound versor still to be named in each order. We may without ambiguity speak of compound five, seven, and eightfold symmetry. Only one other versor needs to be given particular attention. The negative of the

compound fivefold versor describes a group which will be later called the proto-dekahedroid, and it is itself called the dekahedroid distributor, since the name compound decimal would be ambiguous.

21. Resolution of Groups. Not every factor of a transformation will itself be a transformation belonging to the group under discussion. Consider the following perverter in space of three dimensions that may be factored as indicated:

$$\begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{Vmatrix} = \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}.$$

The cyclic group formed from this perverter contains the four following transformations

$$\begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{Vmatrix},$$

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

but neither of the indicated factors appear themselves as members of the group. On the other hand if the following transformation be considered

$$\begin{Vmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{Vmatrix} = \begin{Vmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix},$$

the six members of its cyclic group

$$\begin{Vmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad \begin{Vmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix},$$

$$\begin{Vmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

include both of the factors that we indicated. Because we are studying the rotation groups by means of the quaternion factors of the versors that form the group, it becomes important to distinguish between quaternion versor factors that are themselves members of the group under discussion, and quaternion versor factors that are not themselves members of the group in question when they are taken alone. A quaternion factor that is itself a member of the group under consideration is said to be *free*. A quaternion factor that is not itself a member of the group under consideration is said to be *blended*. It was stated in art. 15 above that to know all the rotation groups it was necessary to know all the possible ways of combining our various quaternion versor groups of different classes. When we speak of combining two given quaternion versor groups of different groups of different classes, we mean that we propose to select the transformations of our final group in such a way that their quaternion factors will be chosen from the given pair of quaternion versor groups. In such a combination the quaternions may be either free or blended in different ways, so that a study of the blending of quaternion versors is necessary. If all the quaternion factors occur free, the final or combination group is said to be *resolved*. If some factors occur free, the group is partly resolved, and the degree of resolution is determined by the lowest power of the quaternion factor that occurs free. For example, if we have a group constructed from the two simple versors HH' and HH'^{-1} then their product $HH'HH'^{-1} = H^2$ must also belong to the group. The group is therefore said to be *semi-resolved*. If H^4 is the lowest power free in some group, then the group is *quarter resolved*.

22. Combination of Cyclic Quaternion Versor Groups. If the result of combining two cyclic quaternion versor groups is to be itself a cyclic group, then where the first power of one quaternion occurs as a factor, the accompanying quaternion factor must bear a power that is prime to its own order, for otherwise the prime powers of the second quaternion will not be reached by

raising the transformation to any power showing that the order of the second quaternion has erroneously been stated too high. Further, by raising to a power, any prime power can be replaced by a first power in the case of either factor we select, but not in the case of both. Hence to examine all the cyclic groups into which H and R' enter as factors, that is, all the cyclic groups that H and R' are capable of producing or yielding, we must examine groups formed from $(HR'^p)^t$ where p takes on different values prime to the order of R' as we pass from group to group, and t takes on successive integral values as the different members of the same group are being developed. We have already seen the more important cyclic groups.

We now wish to establish the theorem that all other quaternion groups made from these same quaternion factors are nothing but resolutions of the cyclic groups.

Let the group under discussion be $(H^pR')^t$, and let it be required to extend the group by including some other versor $H^mR'^n$. Then by virtue of the identity

$$H^mR'^n(H^pR')^{-n} = H^{m-pn}$$

a free quaternion is produced. Hence to get successively all the remaining groups derived from these quaternion versors we continue to introduce step by step lower and lower integral powers of one of the quaternions until the group is completely resolved.

23. Rules for Blending. The following general principles can be formulated:

(*a*) If the orders of the two quaternion factors are prime to each other no blending takes place, for we can release any quaternion by raising the product to the power equal to the order of the other quaternion. We shall then have free a power of the first quaternion that is prime to its order, and can therefore easily obtain the first power, and the group is fully resolved.

(*b*) In blending quaternion versors of odd order, the order of the product will be the lowest common multiple of the orders of the factors.

(*c*) In the case of even versors, if mn is the order of one, np the order of the other, n the highest common factor, then $\frac{1}{2}mnp$ will be the order of the product if m and p are both odd; otherwise the resulting order will be mnp.

Put

$$H^{(1/2)mn} = R'^{(1/2)np} = -E.$$

Then

$$(HR')^{(1/2)mnp} = (-E)^p(-E)^m = (-E)^{m+p}.$$

In other words, an oddly even versor will not blend with an evenly even versor as far as the factor 2 is concerned.

(*d*) If m and p are both prime to n, we have essentially two quaternions to blend of order n, and the quaternions of orders m and p are thrown in in addition without affecting the blending. But even if we do not wish to break the factors up this way, the same number of blended groups will be formed as if we had only quaternions of order n to blend.

(*e*) A self complementary group is always semi-resolved.

$$(HR'^p)(HR'^{-p}) = H^2.$$

(*f*) If $-E$ is not to be free, the free quaternions must be of odd order.

24. Applications. Prime versors. Two third order quaternion versors yield (i.e. choosing quaternion factors from the cyclic groups of two quaternion versors of different classes, there can be formed) a pair of normal or complementary threefold planes in addition to the resolved group. With quaternion versors whose order is not a multiple of three, the third order quaternion versor does not blend. Wherever the order is a multiple of three, there will be possible a pair of complementary cyclic groups in addition to the resolved group.

Two fifth order versors yield a pair of complementary compound versors in addition to the pair of planes and the resolved group. Two seventh order quaternions versors supply four compound versors of the same species. Two eleventh order versors give rise to eight compound versors of two species, four of each species.

Composite Odd Versors. Two ninth order versors give rise to the usual pair of plane versors, the remaining places being filled by versors where the order of one of the quaternion versors has been reduced to three. Upon the addition of H^3 to the cyclic group partial resolution is affected, and although this is the last stage before complete resolution, it is not semi-resolution, so there are possible two complementary 1/3 resolved groups.

Hamilton Units. If the other quaternion versor is of odd, or of oddly even order, there will be only one group possible, and it will be completely resolved. If the order of the other quaternion is a multiple of 8, there will be a semi-resolved group in addition to the fully resolved group. If the order of the other quaternion is an odd multiple of 4, there will be a pair of complementary cyclic groups, and a semi-resolved group in which the two latter are united bringing in $-E$, and in addition the resolved group.

Oddly Even Versors. The Sixth order versor is a negative third order versor. The cyclic groups include two complementary planes and two complementary alternators. In the case of the latter, however, $-E$ is free, and the groups are 1/3 resolved. If the planes are semi-resolved, we obtain what is really a fully resolved third order group. In full resolution the center of symmetry $-E$ is added. Sixth and 8th order versors do not blend for

$$(AZ')^4 = A^4Z'^4 = -A^4 = A.$$

If H is the 12th order versor

$$(AH')^9 = A^9H'^9 = -H'^9 = H'^3$$

the cyclic is itself 1/3 resolved. Besides the complementary cyclic, there is only the fully resolved group. Blending with a 10th order versor, the cyclic is of the 15th order and is semi-resolved.

In the case of the decimal or 10th order versors, the cyclic groups are complementary fivefold planes, complementary alternating groups, complementary compound fivefold versor groups, and the compound decimal groups formed from the negatives

of the compound fivefold versors. The alternators and the compound decimal versors represent 1/5 resolution since there is a center of symmetry present. Semi-resolution is of course unique, without a center, but contains both planes, both compound fivefold versors as well as the free fivefold quaternion versors.

Eightfold Versors. The eight fold versors will not blend with odd or oddly even versors, and with versors that are odd multiples of four they are always semi-resolved, as for example in combination with 4 and 12 fold versors. When two 8 fold versors are combined with each other, four cyclic groups are formed, two fourfold planes, and two fourfold alternators. In quarter resolution the center of symmetry is added, so that a group is obtained embracing a fourfold plane, the normal twofold plane, and the alternator formed by applying the center to the fourfold plane. There are two complementary quarter resolved groups, and in semi-resolution these two are combined due to the addition of two Hamilton Units. Finally in the resolved group there appear, besides the quaternion versors, a pair of complementary groups of quasi-isotropic eightfold symmetry.

Twelvefold Versors. We derive the same number of cyclic groups as in the preceding case. The planes are of sixfold symmetry. The alternators consist each of a twofold plane normal to a threefold plane. The first step in resolution adds a center, and concomitantly a plane normal to the two fold plane. By adding Hamilton Units the group can be 1/3 resolved. Quarter resolution is effected by adding the threefold quaternion versors to the cyclic groups, and introduces a threefold plane normal to the one already present. In the case of the 1/6, 1/4, and 1/3 resolved groups, a complementary group exists. The semi-resolved group is unique, and may be looked upon as uniting the similar complementary groups, or as uniting the 1/4 and the 1/3 resolved groups. The fully resolved group introduces quasi-isotropic sixfold subgroups along with the free quaternion versors. It will be observed that the even steps of resolution can be

described as well by the addition of a simple versor as by the addition of a quaternion versor. Hence such groups are conveniently named as mere combinations of planes. The 1/3 resolved group does not lend itself to such a description, however.

IV. Blending of Whorls

25. General Notions. If a rotation group is to be formed with transformations whose quaternion factors are to be drawn from two whorls, then versors will be formed either from the combination of stem with stem, or of peripheral with peripheral, or of stem with peripheral. Versors formed from the combination of stem with stem, i.e., $H^mH'^n$ are called *principals*. The product of two principals is itself a principal

$$(H^mH'^n)(H^pH'^q) = H^{m+p}H'^{n+q}$$

so that these versors among themselves constitute a subgroup called the principal group. All that we have just said in the preceding part applies to the principal group. The combination of two peripherals with each other is always a simple twofold versor, (because a peripheral is a Hamilton Unit) and is called a *lateral*.

$$(IH^m)(I'H'^n) = II'H^mH'^n.$$

The product of two laterals is a principal

$$(II'H^mH'^n)(II'H^pH'^q) = H^{p-m}H'^{q-n},$$

while the product of principal and lateral is again a lateral

$$(H^mH'^n)(II'H^pH'^q) = II'H^{p-m}H'^{q-n},$$

$$(II'H^pH'^q)(H^mH'^n) = II'H^{p+m}H'^{q+n},$$

so that the principals and the laterals together likewise form a subgroup. In fact we may have this and no more. The product of peripheral and stem is called a *medial*

$$(IH^m)(H'^n) = IH^mH'^n.$$

Except where the stem is itself a Hamilton Unit, the medial will always be a transformation of quasi-isotropic symmetry (cf. art. 20e and 24d). In combining a whorl with a versor that is not attended by peripherals, there will of course be no laterals. If two whorls are used, however, there must be laterals if there are medials. For the peripherals must be present either in medials or in laterals, so that if both stems have peripherals then

$$(IH^mH'^n)(I'H^pH'^q) = II'H^{p+m}H'^{q-n}$$

laterals are obtained by multiplying medials derived from opposite stems. Hence three cases arise. Laterals are found without medials in the combination of two whorls, medials without laterals in the combination of versor and whorl, and the more involved case where all are present.

26. Groups Containing Laterals Alone. Prior to the introduction of laterals, there is nothing to distinguish directions in the planes of rotation. Hence any lateral may be adopted as the first one without loss of generality. We therefore select II'. When this first lateral has been added to the principal group, and it may be added to any of the groups just considered, multiplication by the principal group takes care of the rest. A description of the distribution of the laterals would be a description of the principal group itself. In studying perversions, however, the presence of laterals makes a difference, for the intersections of the lateral planes with the planes of rotation of the principal group determine fixed directions in these planes.

27. Formation of Medials. Peripherals are Hamilton Units, and as we have seen, do not blend with odd versors, or with oddly even versors If the versor to be blended with the Hamilton Unit is an odd power of four, it is 1/4 resolved, and if its order is a multiple of eight it is semi-resolved. Since

$$H^{-m}IH^nH^m = IH^{n+2m},$$

the peripherals fall into two types accordingly as n is even or odd.

If therefore one is free, the whole type is free. If both types are free, then

$$(IH^n)^{-1}IH^{n+1} = H,$$

so the stem is also free and the groups are resolved.

28. Medials without Laterals. If one stem has no peripherals, then the only way to blend the peripherals of the other stem is in the formation of medials.

$$(IH^mH'^n)(IH^{m+1}H'^p) = -HH'^{n+p}.$$

If the order of H' is a multiple of eight H'^2 is free, so if there is to be blending n and p must be odd. If the order is an odd multiple of four H'^2 would have an oddly even order and not blend with a Hamilton Unit. Hence in either case n and p are both odd and $n + p$ is even. But $-H'^2$ is free so that we have released H. The minus sign is only important in case H is oddly even, in which case it is freed in an attempt to blend with an evenly even versor. Thus in any event the blending of all the peripherals means the release of the stem, so that in the absence of laterals we must choose between leaving the stem unblended, and leaving alternate peripherals unblended.

29. Groups formed from Versor and Whorl.

(*a*) Order of Lone Versor not divisible by four.

Only the resolved group.

(*b*) Order of Lone Versor Divisible by Four. Stem Oddly Even.

The peripherals of one type are the negatives of the peripherals of the other. $-E$ always free. The peripherals may be blended with the lone versor, or the group may be resolved.

(*c*) Orders of both Divisible by Four.

The stem may be free and all the peripherals blended. The stem may blend with the lone versor, and the latter will also take up the alternate peripherals.

The group may be resolved.

30. Medials with Laterals. The presence of medials means that the principal group will be semi-resolved, except where the order of the stem is four times an odd number. If the lateral is taken to be II', then we may expect to have for a nucleus all the products of

(*a*) $\qquad E, \quad II', \quad -IHH', \quad HI'H'$

with $H^{2m}H'^{2n}$ if the stem is to be semi-resolved, or with $(-H^2)^m(-H'^2)^n$ if the stems are quarter resolved, for by squaring the medials we release the negative squares of the stems. The nucleus is itself a complete sub-group. Additional blocks of transformations are obtained by multiplying by H, H' and HH'.

$$\begin{array}{cccc} H' & II'H' & IH & HI' \\ H & IHI' & IH' & I'H' \\ HH' & IHI'H' & I & I' \end{array} .$$

By combining these blocks with the nuclear group, the various groups are built up. If both stems are odd multiples of four there is a quarter resolved group consisting of the nuclear group indicated. If a double sign be prefixed to the transformations of the quarter resolved group, we obtain the

(*b*) Reduced Principal Group which exists if both stems are evenly even:

$E, II', IHH', HI'H'$ and products with $H^{2m}H'^{2n}$.

(*c*) The Resolved Medial Group requires the blended stem to be evenly even:

$H', II'H', IH, HI', E, II', IHH', HI'H'$ and products with $H^{2m}H'^{2n}$.

(*d*) The Semi-Resolved Principal requires both stems to be evenly even:

$HH', IHI'H', I, I', E, II', IHH', HI'H'$ and products with $H^{2m}H'^{2n}$.

(*e*) In addition to these there remains the fully resolved group.

V. Perverting Whorls

31. General Forms for Whorl Pervertors. Any perverter can be written in the form $\Pi RS'$. The effect of such a perverter upon the stem of a whorl when applied to it as a converter is as follows:

$$(\Pi RS')^{-1}H(\Pi RS') = S'^{-1}H'S'$$
$$(\Pi RS')^{-1}H'(\Pi RS') = R^{-1}HR$$

and similarly for I and I'. To prevent the introduction of new quaternions we have the conditions:

$$S^{-1}HS = H^m \qquad \text{and} \qquad R^{-1}HR = H^n$$

and by expanding H into the form $E \cos q + K \sin q$ we may show that m and n must be equal in absolute value to unity. If we suppose that $n = 1$

$$R^{-1}HR = H,$$
$$HR = RH.$$

But H is only commutative with quaternions of its own class that are powers of itself, so we may replace R in this case by a power of H. The second alternative

$$R^{-1}HR = H^{-1}$$

may be rewritten

$$R^{-1}(E \cos q + K \sin q)R = E \cos q - K \sin q,$$

whence

$$R^{-1}KR = -K,$$
$$R = KRK.$$

Put

$$R = wE + xI + yJ + zK$$

and obtain

$$wE + xI + yJ + zK = -wE + xI + yJ - zK$$

from which

$$w = 0 \qquad \text{and} \qquad z = 0$$

so that

$$R = xI + yJ$$

must be a Hamilton Unit peripheral to K. A similar treatment applies to S, so that all admissible perverters may be expressed in the forms

$$\Pi H^m H'^n, \quad \Pi I H^m H'^n, \quad \Pi I' H^m H'^n, \quad -\Pi I I' H^m H'^n.$$

32. Perverted Stems. It will be convenient to replace $-\Pi II'$ with the single symbol Π^*, and to write $\Theta^\mu \Theta^{*\nu}$ instead of $H^m H'^n$ where

$$\begin{aligned} m &= \mu + \nu \qquad \mu = \tfrac{1}{2}(m+n), \\ n &= \mu - \nu \qquad \nu = \tfrac{1}{2}(m-n). \end{aligned}$$

The simple rotations Θ and Θ^* convert the perverters Π and Π^* as follows:

$$\begin{aligned} \Theta^{-m}\Pi\Theta^m &= \Pi, \\ \Theta^{-m}\Pi^*\Theta^m &= \Pi\Theta^{2m}, \\ \Theta^{*-m}\Pi\Theta^{*m} &= \Pi\Theta^{*2m}, \\ \Theta^{*-m}\Pi^*\Theta^{*m} &= \Pi^*. \end{aligned}$$

The axis of perversion of Π therefore lies in the stator plane of Θ, while the axis of Π^* lies in the stator plane of Θ^*. Our perverters can be written

$$\begin{aligned} \Pi\Theta^\mu\Theta^{*\nu} &= \Theta^{*-(1/2)\nu}\Pi\Theta^{*(1/2)\nu}\Theta^\mu, \\ \Pi^*\Theta^\mu\Theta^{*\nu} &= \Theta^{-(1/2)\mu}\Pi^*\Theta^{(1/2)\mu}\Theta^{*\nu}, \end{aligned}$$

showing that one of the indices gives the position of the axis of preversion while the other gives the exponent of the simple versor out of which the given compound perverter may be considered to have been derived. We call these perverted stems. To correlate the positions with the lateral planes

$$(II'\Theta^m\Theta^{*n})^{-1}\Pi\Theta^{*\nu}(II'\Theta^m\Theta^{*n}) = \Theta^{*2n-\nu},$$

so that if $2n - \nu = \nu$, that is if $n = \nu$ the perverter will be transformed into itself. Hence the axis of the perverter lies in the sheaf of planes $II'\Theta^m\Theta^{*\nu}$. Similar considerations apply to the other perverters.

FORMULAS FOR USE WITH WHORLS

$$\Theta = HH' = \Theta^* H'^2 = \Theta^{*-1} H^2,$$
$$\Theta^* = HH'^{-1} = \Theta H'^{-2} = \Theta^{-1} H^2,$$
$$H = \Theta^{1/2}\Theta^{*1/2} = \Theta H'^{-1} = \Theta^* H',$$
$$H' = \Theta^{1/2}\Theta^{*-(1/2)} = \Theta H^{-1} = \Theta^{*-1} H,$$
$$HI = IH^{-1} = H^{1/2} I H^{-(1/2)},$$
$$I\Theta = \Theta^* I, \qquad I\Theta^* = \Theta^{-1} I,$$
$$I'\Theta = \Theta^* I', \qquad I'\Theta^* = \Theta I',$$
$$II'\Theta = \Theta^{-1} II', \; II'\Theta^* = \Theta^{*-1} II',$$
$$\Pi^* = -I\Pi I = -I'\Pi I' = -\Pi II' = -II'\Pi,$$
$$\Pi\Theta = \Theta\Pi, \qquad \Pi^*\Theta^* = \Theta^*\Pi^*,$$
$$\Pi\Theta^* = \Theta^{*-1}\Pi = H^{-1}\Pi H = H'\Pi H'^{-1} = \Theta^{*-(1/2)}\Pi\Theta^{*1/2},$$
$$\Pi^*\Theta = \Theta^{-1}\Pi^* = H^{-1}\Pi^* H = H'\Pi^* H'^{-1} = \Theta^{-1/2}\Pi^*\Theta,$$
$$\Pi I = I'\Pi = \Pi^* I' = I\Pi^*,$$
$$\Pi^* I = I\Pi = \Pi I' = I'\Pi^*.$$

See also article 19.

33. Perverting a Pair of Planes. As the perverters we are considering interchange the quaternion versors used as stems, we require that the quaternion groups involved be similar, and that the principal group be reversible, or that the diagram of the principal group be symmetrical to its diagonals. The examination of the squares of the perverters reveals other conditions.

$$(\Pi\Theta^\mu\Theta^{*\nu})^2 = \Theta^{2\mu},$$
$$(\Pi^*\Theta^\mu\Theta^{*\nu})^2 = \Theta^{*2\nu},$$

so the existence of these planes is also required. Let us suppose that the plane Θ exists, as well as the plane $\phi^x = \Theta^*$, so that the principal group will contain $\Theta^x\phi^y$ for all integral values of x and y. We have then the conditions imposed by the principal group that $(\Theta^\mu)^2 = \Theta$ or Θ^2 and that $(\Theta^{*\nu})^2 = \phi$ or ϕ^2 for the two perverters respectively. The laterals if present impose a similar condition on the location of the axis in each case:

$$(\Pi\Theta^\mu\Theta^{*\nu})^{-1} II' (\Pi\Theta^\mu\Theta^{*\nu}) = II'\Theta^{2\mu}\Theta^{*2\nu},$$
$$(\Pi^*\Theta^\mu\Theta^{*\nu})^{-1} II' (\Pi^*\Theta^\mu\Theta^{*\nu}) = II'\Theta^{2\mu}\Theta^{*2\nu},$$

so that the perverters finally allowed reduce to

$$\Pi,\ \Pi\theta^{1/2},\ \Pi\phi^{1/2},\ \Pi\theta^{1/2}\phi^{1/2},$$

with corresponding forms derived from Π^*. If we call positions *in* the lateral planes *primary*, and positions in *between* the planes *secondary*, then we may distinguish primary and secondary perverters, though the distinction is not satisfactory in resolved groups.

34. Reduction to Lowest Terms. In determining the various species of perverters introduced into a group, the following reductions simplify the discussion.

$$(\Pi\theta)^2 = \theta^2,$$

and therefore if m is an odd number we may write the following identities:

$$(\Pi\theta)^m = \Pi\theta^m,$$
$$(\Pi\theta^2)^m = \Pi\theta^{2m},$$
$$(\Pi\theta^4)^m = \Pi\theta^{4m},$$

so that if θ is of odd order, we can express all perverters of the form $\Pi\theta^m$ as powers of a single perverter $\Pi\theta$, while if the order of θ is oddly even we need two $\Pi\theta$ and $\Pi\theta^2$, if the order of θ is an odd multiple of four, three are required $\Pi\theta$, $\Pi\theta^2$, and $\Pi\theta^4$ etc.

35. Correlation of Perverters. In the absence of laterals there are no absolute directions and so we have in general four perverters to introduce

$$\Pi,\ \Pi^*,\ \Pi\theta^{1/2},\ \text{and}\ \Pi^*\phi^{1/2},$$

or in the case of a single plane

$$\Pi,\ \Pi\theta^{1/2},\ \Pi^*,\ -\Pi^*.$$

In the presence of laterals multiplication correlates the perverters:

$$II'\Pi = -\Pi^*,$$

and in general

$$II'\Pi\theta^\mu\theta^{*\nu} = -\Pi^*\theta^\mu\theta^{*\nu},$$

so that the perverters of the two series are thus correlated, and the number of perversions is again reduced to four. The negative sign is incorporated into the versors by putting

$$H^\gamma = K.$$

If H is odd γ is a fraction, while if H is of even order γ is integral.

$$-E = \Theta^\gamma\Theta^{*\gamma},$$
$$-\Pi\Theta^\rho\Theta^{*\rho} = \Pi\Theta^{\rho+\gamma}\Theta^{*\rho+\gamma},$$

so that the axis is turned through 90°. If both planes are odd Π^* is derived from $\Pi\Theta^{1/2}\phi^{1/2}$, $\Pi^*\Theta^{1/2}$ from $\Pi\phi^{1/2}$, $\Pi^*\phi^{1/2}$ from $\Pi\Theta^{1/2}$ and $\Pi^*\Theta^{1/2}\phi^{1/2}$ from Π. On the other hand with both planes even Π^* is derived from Π, $\Pi^*\Theta^{1/2}$ from $\Pi\Theta^{1/2}$, $\Pi^*\phi^{1/2}$ from $\Pi\phi^{1/2}$, and $\Pi^*\Theta^{1/2}\phi^{1/2}$ from $\Pi\Theta^{1/2}\phi^{1/2}$. If we are speaking of a self-complementary group such as a pair of similar planes the four perverters will not produce dissimilar groups. For odd pairs of similar planes we should drop the compound secondary perverter $\Pi\Theta^{1/2}\Theta^{*1/2}$, while for pairs of similar even planes, we should drop the compound primary perverter, $\Pi\Theta^{1/2}$. In the case of fully resolved groups correlation proceeds further.

$$H = \Theta^{1/2}\Theta^{*1/2},$$

so that $\Pi\Theta^{1/2}\Theta^{*1/2}$ will be formed from Π by multiplication, while $\Pi\Theta^{1/2} = \Pi\Theta^{*1/2}H$ will be formed from $\Pi\Theta^{*1/2}$, and moreover, as there are lateral planes in both positions, there will be no point in distinguishing these.

36. Perverted Peripherals. In the case of the perverters just considered the stems play the principal part. We neglected the considerations of perverters of the form $\Pi IH^mH'^n$ and $\Pi I'H^mH'^n$ which we call *perverted peripherals*. The perverters may be rewritten as conversions of ΠI

$$\Pi IH^mH'^n = (\Theta^{(1/2)m}\Theta^{*-(1/2)n})^{-1}(\Pi I)(\Theta^{(1/2)m}\Theta^{*-(1/2)n}),$$

and similarly for the other perverters. Hence we are here con-

cerned with only one species of transformation. Taking as usual

$$X = \frac{E + I}{\sqrt{2}},$$

we have

$$\Pi I = \Pi X^2 = (X^{-1}\Pi X)(XX'),$$
$$\Pi I' = \Pi X'^2 = (X\Pi X^{-1})(XX').$$

The axes, therefore, lie in the plane of the stator of II', and half way between the rotor planes of the principal group. The squares of the perverters indicate the presence of laterals:

$$(\Pi I H^m H'^n)^2 = II'\Theta^m\Theta^{*-n},$$
$$(\Pi I' H^m H'^n)^2 = II'\Theta^n\Theta^{*m}.$$

The conversion test applied to the principal group

$$(\Pi I H^p H'^q)^{-1} H^m H'^n (\Pi I H^p H'^q) = H^{-n} H'^m,$$
$$(\Pi I' H^p H'^q)^{-1} H^m H'^n (\Pi I' H^p H'^q) = H^n H'^{-m},$$

shows that the exponents of the quaternions are interchanged with a single reversal of sign. This is equivalent to turning the graph of the principal group through 90°. A principal group that admits such a conversion we decided to call quadrantal (Cf. Art. 19). The resolved and semi-resolved groups are obviously quadrantal, but some cyclic groups in particular have the property. The first twelve of these with prime orders are listed. They were obtained as follows:

Let

$$H^p = E,$$

and let

$$(H^m H'^n)^s = H^{-n} H'^m,$$

whence

$$mz = px - n \qquad \text{and} \qquad nz = py + m,$$

so that

$$p(nx - my) = m^2 + n^2.$$

As all these quantities are integers, we conclude that p the order of the given quaternion must be either a factor or a multiple of $m^2 + n^2$.

TABLE OF QUADRANTAL CYCLIC GROUPS

Order	Versor	Order	Versor
5	HH'^{3}	53	HH'^{30}
13	HH'^{8}	61	HH'^{11}
17	HH'^{4}	73	HH'^{27}
29	HH'^{17}	89	HH'^{55}
37	HH'^{6}	97	HH'^{71}
41	HH'^{9}	101	HH'^{10}

Others of composite order can be formed from these.

A general consideration of the number of each species that can be made from a given pair of quaternion versors would lead us to suspect the presence of quadrantal species if the order is $4n + 1$ where n is an integer and the order is prime, for apart from the quaternion versor H and the planes HH' and HH'^{-1}, we expect four of each species, and but two of a quadrantal species. If the order is not prime, there will be other cyclic versors not expressible in the form HH'^{n}, and so the argument does not cover these cases.

If the laterals are converted we may show that in general two series of perverters may be introduced, ΠI and $\Pi I\Omega^{1/2}$ just as we showed something of the same nature in the case of perverted stems, and this means that the perverters will be inserted in alternate lateral planes. As prior to the introduction of perverters we had no reason to prefer one lateral to another, the two methods of perversion will be expected to give similar results. We may conclude therefore that quadrantal principal groups (including all resolved and semi-resolved principal groups) having laterals attached admit perverted peripherals in two ways that do not differ specifically from each other.

37. Perversion of Groups Containing Medials. It is only a question of perverting groups containing both medials and laterals, because the two quaternion groups must be similar. Perverted peripherals need not be considered especially, because they will be produced from perverted stems by intermultiplication with the medials, and in the introduction of perverted stems no new principles occur. It can be shown by vector analysis that the rotor planes of the medials intersect in pairs in the lateral

planes in lines half way between the intersections of the principal rotor planes, and we have seen that the axes of perverted stems when introduced also fall on these same medial lines. We proceed to the discussion of the various genera of groups that fall under this division.

(*a*) Perversion of the Quarter Resolved Group. In this group the quaternions $-H^2$ and $-H'^2$ are free. The planes are Θ^2 and Θ^{*2}, and these planes are of odd order. The species of perversion then are effected with Π, $\Pi\Theta$, $\Pi\Theta^*$ and $\Pi\Theta\Theta^*$.

(*b*) Perversion of Group with Reduced Principal. In this group the planes are again Θ^2 and Θ^{*2}, but H^2 is free, and $\Theta = \Theta^* H^2$, so that we have a primary perverter Π and a secondary perverter $\Pi\Theta^*$. The stems are evenly even, so that these two methods are specifically different.

(*c*) Perversion of Medial Group with Semi-Resolved Principal. The stems are evenly even, and the planes are Θ and Θ^* and are both even. It will be found that the perverters with fractional indices convert the medials present into other medials that have been excluded from the group, so that there are only two admissible systems of perversion, with Π and with $\Pi\Theta^{1/2}\Theta^{*1/2} = \Pi H$, the former being primary and the latter secondary.

(*d*) Perversion of Fully Resolved Group. For this case, as we saw above, there are two methods of perversion, with Π and with $\Pi\Theta^{*1/2}$, but these two methods do not differ specifically from each other.

VI. The Blending of Umbels

38. Principles. There are two new principles that come into the blending of umbels. The first is the matter of type. All the quaternions of the same type must be in the same state. In particular, a free tetartoidal releases all the others. But with the tetartoidals free the tetartoid umbel can be reconstructed. Hence a free tetartiodal releases the tetartoid umbel. Similarly, a free gyroidal releases the gyroid umbel, a free cardinal releases the gyroid umbel, and any free quaternion releases the icosa-

hedral umbel. The second principle has to do with the matter of sub-groups. The group consisting of the sphenoidals alone is expanded into the tetartoid umbel by the addition of a single tetartoidal, and the tetartoid umbel may be expanded into the gyroid umbel by the addition of a single cardinal, or a single gyroidal. As a consequence, if a sphenoidal is free, then whatever blends with one tetartoidal must blend with every tetartoidal, and if a tetartoidal is free, then whatever is combined with any one cardinal or with anyone gyroidal must be combined with every cardinal and every gyroidal. These principles are easily verified by an inspection of the tables.

39. Blending of Umbel with Versor. (*a*) A versor of the third order is capable of blending with the tetartoidals which are versors of the sixth order, but then

$$(HA')(HB')^{-1} = -I,$$

so that the sphenoidals are released. (*b*) If the order of the lone versor is not a multiple of three, there will be no blending and only a resolved group will be obtained. (*c*) In like manner if a single versor is to blend with a gyroid umbel, its order must be a multiple of four, and it will be semi-resolved, for the product of two cardinals is in general a tetartoidal.

40. Blending of Umbel with Whorl. (*a*) No whorl can be blended with the tetartoid umbel, for the sphenoidals will always be freed by the multiplication indicated above. They will not then blend the peripherals, and these being free supply their own stem. (*b*) A whorl the order of whose stem is divisible by three may blend the tetartoidals in a gyroid umbel. In the absence of a free tetartoidal there is no necessity for cardinals and gyroidals always to blend with the same versor, so that it is possible for them to take up the peripherals. Take for example the whorl

$$A, \text{ etc.}, F, -P, N, -F, P, -N.$$

The stem A will blend with the tetartoidals

$$AA', AB', AC', AD', \text{ etc.},$$
$$I', J', K' \text{ free},$$
$$FX', FX'^{-1}, FG', FF',$$
$$PZ'^{-1}, PP', PQ', PZ',$$
$$NM', NY'^{-1}, NN', NY'.$$

A double sign is to be understood in every case as $-E = I'^{2}$ is free.

41. Substitution of Umbel for Lone Versor. If in the gyroid umbel the tetartoid umbel is free, then we need only consider one gyroidal say F', because all gyroidals and cardinals will be multiplied into the same quaternions. We may now use the methods for the blending of versor and whorl. (*a*) If the stem of the whorl is a multiple of four, it can be blended and alternate peripherals be released:

Take for the whorl

$$Z, \text{ etc.}, I, P, J, Q, \text{ etc.}$$

The stem is to be blended:

$$ZF', K, Z^{3}F',$$

alternate peripherals to be free:

$$P, Q,$$

the rest to be blended:

$$IF' \text{ and } JF',$$

a double sign being understood. Multiply through by the tetartoid umbel and obtain the required group. (*b*) Another way is to blend all the peripherals and allow the stem to go free:

Free:

$$Z, K, Z^{3}, E.$$

Medials:

$$IF', PF', JF', QF'.$$

Multiply through by the tetartoid umbel and obtain the required group. The order of the stem does not affect this manner of blending.

42. Blending of Two Tetartoid Umbels. If the two tetartoidals A and B are to be blended with different versors, then these must be chosen from a tetartoid umbel of the other class, and we obtain (*a*) a group of collinear planes (the collinear tetartoid)

$$AA', BB', CC', DD', II', JJ', KK', E.$$

(*b*) A center of symmetry may be added by prefixing a double sign to each of these transformations. This operation semi-resolves the Hamilton Units.

(*c*) The three sphenoidals may now be released, causing a general blending of like powers of the tetartoidals with each other.

(*d*) Finally the umbels can be completely released.

43. Blending of Unlike Umbels. When we attempt to blend a tetartoid with a gyroid umbel, we find the sphenoidals of the tetartoid umbel will not be able to take care of the cardinals. Whether two cardinals blend with the same Hamilton Unit

$$(IX')(IY') = A'^2,$$

or with different Hamilton Units

$$(IX')(JY') = KA'^5,$$

in either case a tetartoidal will be released and this in turn releases the tetartoid umbel and resolves the group. There can of course be no question of blending an icosahedral umbel with anything else except a similar umbel, because only an icosahedral umbel has both six and tenfold versors.

44. Blending Gyroid Umbels. It might be thought possible to blend two gyroid umbels by allowing the gyroidals of one to blend the cardinals of the other and vice versa. But since $(IX')^2 = I'$, the cardinals being of order eight and the gyroidals of order four, semi-resolution takes place and the sphenoidals are free. The product of a cardinal and a sphenoidal not a power of itself is a gyroidal, and the product of a gyroidal and a sphenoidal not peripheral to it is a cardinal, so that in this way car-

dinals and gyroidals can be interchanged by multiplication by the free sphenoidals, and we are combining cardinal with cardinal after all.

(*a*) Blending, therefore, cardinal with cardinal, we obtain a group of collinear planes, (the collinear gyroid) consisting of

$$XX', YY', ZZ', FF', GG', PP', QQ', NN', MM'$$

in addition to the planes of the collinear tetartoid group.

(*b*) Or we may choose to blend in alternating symmetry. This replaces the first three planes by the alternators $-XX'$, $-YY'$, $-ZZ'$, and the next six by their normal or completely perpendicular planes. The sub-group of collinear tetartoid planes remains as before.

(*c*) Adding a center of symmetry not only combines these two, but now for the first time introduces the negatives of the collinear tetartoid versors.

(*d*) If we semi-resolve the cardinals and release the sphenoidals, we obtain the rotation group of the regular octahedroid. In this group the tetartoidals are still blended.

(*e*) The tetartoidals may be released, swelling the number of transformations up to 576. This is the rotation group of the 24hedroid.

Finally the gyroid umbels themselves may be released giving us the rotation group of the 48hedroid.

45. Blending of the Icosahedral Umbel. As the presence of a free quaternion releases the icosahedral umbel, it can only be blended with a similar umbel, but as in the case of the gyroid umbel, it can do so in two ways. (*a*) One way leads to a collinear group of 60 transformations, but (*b*) the decimal versors can also blend in compound symmetry instead of yielding planes, and if this is done we obtain the group of rotations belonging to the pentahedroid. (*c*) and (*d*) To either of these groups a center may be added, and then (*e*) resolving the group a rotation group of 7200 transformations is obtained, appearing in the symmetry of the 120- and 600hedroids.

VII. Perverting an Umbel

46. The rotation groups that are to be considered here will necessarily be derived from two similar umbels. Let us suppose a perverter to be introduced of the form $\Pi RS'$. If we apply the conversion test

$$(\Pi RS')^{-1}H(\Pi SR') = S'^{-1}H'S',$$
$$(\Pi RS')^{-1}H'(\Pi RS') = R^{-1}HR,$$

we see that the quaternions must be converted into similar quaternions of the other class. If we replace H with I, J, K respectively, we inquire what converters will convert the group of sphenoidals into itself. There will be 24 such converters, and if we examine the conversion table for the gyroid umbel, we shall find that all 24 converters have already been found. Hence for the tetartoid and gyroid umbels, R and S must be taken from the gyroid umbel. Similarly, if we examine the conversions of the decimal versors in the tables for the icosahedral umbel, we shall find that for the icosahedral umbel, R and S must be chosen from the umbel itself, there being but 12×5 ways of converting the decimals versors into one another.

47. Types of Simple Perverters. The types will be determined largely by the presence of planes. Consider therefore the conversion

$$(AA')^{-1}R^{-1}\Pi R(AA') = (A^{-1}RA)^{-1}\Pi(A^{-1}RA).$$

From this we conclude that the types as established by the planes will correspond to the types of the quaternion versor R itself. In the case of the pentahedroid the planes take the form $Y'AA'Y'^{-1}$, so we convert back and forth by means of Y'.

(*a*) Types Π and $-\Pi$. These perverters may be applied to any rotation group except that of the pentahedroid, because of the Y' twist in the latter. They will not yield different perversion groups if $\div E$ is present.

(*b*) Type $-I\Pi I$. From the equation $-I\Pi I = -\Pi II'$, we conclude that this perverter will be included under $-\Pi$.

(*c*) Type $-F\Pi F$. If we convert RS' with this, we obtain

$$(F^{-1}\Pi F)^{-1}RS'(F^{-1}\Pi F) = (F^{-1}SF)(F'R'F'^{-1}),$$

and an inspection of the tables of conversions shows that apart from the pentahedroid, this perverter can be admitted to any umbel. If we apply the conversion $Y'(\ )Y'^{-1}$

$$Y'(F^{-1}\Pi F)Y'^{-1} = C^{-1}\Pi C,$$

we obtain a perverter for the pentahedroid. In the presence of blended gyroidals $F^{-1}\Pi F$ will be included under

$$(F^{-1}\Pi F)(FF') = -\Pi,$$

and so the perverter will not be listed separately for the gyroid umbel. Since $F^{-1}(\ )F$ converts the resolved tetartroid into itself, it will not be necessary to consider $-F\Pi F$ for the latter group.

(*d*) Type $A^{-1}\Pi A$. If we examine the effect of this perverter by means of the table of conversions, it will be seen that non-collinear planes are introduced, BB' becoming DC'. But $(DC')(CC')^{-1} = -I$, so that the sphenoidals will be free. It will therefore only be tried in the tetartoid umbel when the sphenoidals are free, but not when the tetartoidals are free, for then it can be derived from Π. In like manner it converts XX' into YZ', and this if multiplied by $(ZZ')^{-1}$ means the release of a tetartoidal, so that for the gyroid umbel it will always be included under Π. It cannot be used with the icosahedral umbel without implying full resolution. Since $A^{-1}(\ )A$ converts the tetartoid with released sphenoidals into itself, it will not be necessary to consider $A^{-1}\Pi A$ for any group.

(*e*) Type $X^{-1}\Pi X$. This perverter in like manner implies the presence of free sphenoidals, and in their presence, of course, can be derived from $-F\Pi F$, so this perverter need never be specially considered.

(*f*) Type $W^{-1}\Pi W$. This perverter is only suggested for the icosahedral umbel, but if its effect is studied in the same way, it will be found to release a quaternion and resolve the group.

48. Pervertors of Higher Order. The only perverters of higher order that are to be considered will be those of the forms $R^{-1}\Pi RI$, $R^{-1}\Pi RF$, and $R^{-1}\Pi RX$ for the others can be reduced to simple perverters by raising to suitable powers. In the presence of AA', the perverter $R^{-1}\Pi RI$ can be replaced by a simple perverter:

$$(R^{-1}\Pi RI)(AA') = (A^{-1}R)^{-1}\Pi(A^{-1}R)(R^{-1}ARB),$$

and a simple perverter can be obtained from the latter by cubing. Hence it is only in connection with the gyroid umbel that such perverters need be considered. The location of the axis need not be specially considered for

$$(R^{-1}\Pi RS)^2 = S(RSR^{-1})',$$
$$R^{-1}\Pi RS = (RS)R'^{-1},$$

so that S and RS and therefore R must belong to the gyroid umbel. Then

$$(R^{-1}\Pi RS)RR' = \Pi(RSR),$$

so that whatever perverters are to be considered will be covered by ΠF and ΠX. AA' is converted by the first of these into $A^{-1}A'$ and by the second into $B^{-1}A'$ and multiplication by $(AA')^{-1}$ releases a tetartoidal. As the tetartoid umbel is free and $FK^{-1} = X$ so that ΠX will be covered by ΠF. The result is that we have only one perverter to consider, and only one group to apply it to, for in a fully resolved group ΠF will be covered by Π. The group to be perverted is the rotation group of the regular 24hedroid.

49. Recapitulation.

Quaternion Versor Groups.

Cyclic groups derived from quaternion versors of various orders (Art. 3).

Whorls with stems of various (even) orders (Art. 4).

Umbels: tetartoid, gyroid, icosahedral (Arts. 5, 6, and 7).

Versor with Versor.

Cyclic groups as determined by the rules for blending (Arts. 20 and 24).

Groups formed by the gradual resolution of these (Art. 24).
Perverters for reversible groups: Π, Π*, $\Pi\Theta^{1/2}$, $\Pi^*\Phi^{1/2}$ (Art. 35).
Perverters for self complementary reversible groups: Π and $\Pi\Theta^{1/2}$ (Art. 35).
Perversion of resolved group with Π (Art. 35).

Versor with Whorl (Art. 29).

Blended stem and alternate peripherals free (if orders of both lone versor and of stem are divisible by four)(Art. 29*c*).
Blended peripherals (if order of lone versor is divisible by four) (Art. 29*b*, *c*).
Resolved group (always possible) (Art. 29*a*, *b*, *c*).

Whorl with Whorl, without Medials.

Rotation groups formed by the addition of a lateral plane to the groups formed by the combination of versor with versor (Art. 26).
Perverters for reversible groups with laterals: Π, $\Pi\Theta^{1/2}$, $\Pi\Phi^{1/2}$, $\Pi\Theta^{1/2}\Phi^{1/2}$ (Art. 33).
For self-complementary odd planes with laterals: Π, $\Pi\Theta^{1/2}$, $\Pi\Theta^{*1/2}$ (Art. 35).
For self-complementary even planes with laterals: Π, $\Pi\Theta^{*1/2}$, Π*H* (Art. 35).
For quadrantal groups: Π*I* (Arts. 19 and 36).
For resolved groups: Π (Art. 35).

Whorl with Whorl with Medials (Art. 30).

Quarter resolved principal (both stems odd multiples of four) (Art. 30*a*).
Perverted by Π, ΠΘ, ΠΘ* and ΠH^2 (Art. 37*a*).
Reduced principal (both stems evenly even) (Art. 30*b*).
Perverted by Π, ΠΘ* (Art. 37*b*).
Resolved medial (blended stem evenly even) (Art. 30*c*).
Semi-resolved principal (both stems evenly even) (Art. 30*d*).
Perverted by: Π and Π*H* (Art. 37*c*).
Resolved group (Art. 30*e*).
Perverted by Π alone (Art. 37*d*).

Versor with Umbel (Art. 39).

With tetartoid if order of lone versor is a multiple of three (Art. 39*a*).
With gyroid if order of lone versor is a multiple of four (Art. 39*c*).
Resolved groups.

Whorl with Umbel (Art. 40).

Stem divisible by three blending tetartoidals in gyroid umbel (Art. 40*b*).
Stem divisible by four blended, alternate peripherals free (Art. 41*a*).
Peripherals blended with cardinals and gyroidals (Art. 41*b*).
Resolved groups.

Unlike Umbels (Art. 43).

Tetartoid with gyroid.
Tetartoid with icosahedral.
Gyroid with icosahedral.

Tetartoid with Tetartoid (Arts. 42 and 47).

Collinear Tetartoid (1) may be perverted with
(2) Π, (3) $-\Pi$, (4) $F^{-1}\Pi F$, (5) $\Pi FF'$.
Central tetartoid (6) may be perverted with
(7) Π, (8) $F^{-1}\Pi F$.
Tetartoid with released sphenoidals (9) admits
(10) Π, (11) $F^{-1}\Pi F$.
Resolved tetartoids (12) admit
(13) Π.

Gyroid with Gryoid (Arts. 44 and 47).

Collinear gyroid (14) admitting
(15) Π and (16) $-\Pi$.
Alternating gyroid (17) admitting
(18) Π and (19) $-\Pi$.
Central gyroid (20) admitting
(21) Π.
Octahedroidal rotation group (22)
admits (23) Π.
24-hedroidal rotation group (24) admitting
(25) Π and (26) ΠF (Art. 48).

Group of resolved gyroids (27)
admits (28) Π.
Icosahedral Umbels (Art. 45 and 47).
Collinear icosahedral rotation group (29)
admits (30) Π and (31) $-\Pi$.
Pentahedroidal rotation group (32)
admits (33) $C^{-1}\Pi C$ and (34) $-C^{-1}\Pi C$.
Central icosahedral rotation group (35)
admits (36) Π.
Central pentahedroidal rotation group (37)
admits (38) $C^{-1}\Pi C$.
Hexakosiohedroidal rotation group (39)
admits (40) Π.

VIII. The Pyramids with Polyhedral Bases

50. Groups in Three Space. In discussing the pyramids in crystallography,* we are speaking of groups with a principal axis of manifold symmetry. If there are no lateral planes or axes of symmetry, we have the third order forms. If there is a principal plane of symmetry perpendicular to the principal axis we have double ended forms called bi-pyramids. The single ended forms are sometimes called hemi-pyramids. If there are lateral planes but not lateral axes we obtain di-polygonal hemi-pyramids, but if the lateral planes intersect the principal plane of symmetry in lateral twofold axes the di-polygonal bi-pyramid is formed. Lateral axes without the planes give us the various trapezohedra, or with axes half way between the planes we obtain the scalenohedra. The bi-pyramids are considered to be formed from the addition of a plane and not the addition of a center, because the double ended pyramid that results from adding a center to the third order trigonal pyramid is called a third order rhombohedron, and not a third order bi-pyramid. (The actual transformations of each of these groups are given in detail in

* Cf. Moses and Parsons, Mineralogy, Crystallography and Blow Pipe Analysis.

articles 65 to 70.) We are about to enter into a discussion of analogous groups of symmetry in space of four dimensions.

51. Polyhedral Bases. We might try to be general and not restrict the discussion to pyramids with polyhedral bases, but the purpose of the discussion is to arrive at a better understanding of some of the more complicated groups, and where the base is not polyhedral we get a distorted view of the group that is more complicated than the notion of a group with a pair of principal rotor planes. We do not therefore build pyramid on pyramid. In discussing the polyhedral bases, the collinear tetartoid group is taken as a block of 12 transformations and called block E. by the application of the prefactors $\Pi I I'$, $\Pi F F'$, FF' and their negatives, seven other blocks each of 12 transformations are formed, denoted by their respective prefactors. The five groups of symmetry of the isometric system are described in crystallography comprise the following blocks:

Symmetry	Blocks
Tetartoid	E alone.
Diploid	E and $-\Pi I I'$
Hextetrahedron	E and $-\Pi F F'$
Gyroid	E and FF'
Hexoctahedron	E, $-\Pi I I'$, $-\Pi F F'$, and FF'

To complete the list we ought to add in the icosahedral groups that do not occur in crystallography. Let us call the rotation group of the regular icosahedron the gyro-icosahedral group. It is to be obtained by applying the prefactors $(WW')^t$ to block E. If we apply $-\Pi F F'$ as prefactor to the latter, or $(JW')^{-1}\Pi(JW')$ as prefactor to block E, we obtain the remaining transformations necessary to get the dodecahedron or icosahedron.

52. Hemipyramids. The transformations just given leave one principal direction unaffected, and so we have seven hemipyramids:

Tetartoid ½Pyramid	(1) of Article 49
Diploid ½Pyramid	(3)
Hextetrahedral ½Pyramid	(4)
Gyroid ½Pyramid	(14)
Hexoctahedral ½Pyramid	(16)
Gyro-icosahedral ½Pyramid	(29)
Regular icosahedral	(31)

53. Bi-pyramids. If we add the perverter Π to each of the above groups we obtain a series of bi-pyramids:

Base	Blocks Added	Number (Art. 49)
Tetartoid	$\Pi II'$	(2)
Diploid	$\Pi II'$, $-E$	(7)
Hextetrahedron	$\Pi II'$, $-FF'$	(18)
Gyroid	$\Pi II'$, $\Pi FF'$	(15)
Hexoctahedron	$\Pi II'$, $-E$, $-FF'$, $\Pi FF'$	(21)
G-icosahedral	$-(JW')^{-1}\Pi(JW')$	(30)
Icosahedron	$-(JW')^{-1}\Pi(JW')$ and $-(WW')^{i}$	(36)

54. Central Pyramids. Double ended pyramids can also be formed by the addition of the center $-E$. It will be seen that some of the groups can be described in more than one way.

Base	Blocks Added	Group Number (Art. 49)
Tetartoid	$-E$	(6)
Diploid	$-E$, $\Pi II'$	(7)
Hextetetrahedron	$-E$, $\Pi FF'$	(8)
Gyroid	$-E$, $-FF'$	(20)
Hexoctahedron	$-E$, $\Pi II'$, $\Pi FF'$, $-FF'$	(21)
G-icosahedral	$-(WW')^{i}$	(35)
Reg. Icosahedron	$-(WW')^{i}$, $-(JW')^{-1}\Pi(JW')$	(36)

55. Alternating Pyramids. There are still some groups having

a single principal axis that have not been described. But there remain still two more ways of producing double ended pyramids. Half of the blocks may be replaced by their opposites or negatives, and the other half retained as they are. Or we may replace half with their reflections or their products with Π, retaining the other half as they are. In trying this on the hexoctahedron we find three ways of selecting the blocks for alternation, two ways of alternating, and with the results equal in pairs there come out three alternating forms, two of which have been previously described.

Base	Method	Blocks	Group Number (Art. 49)
Diploid	$-E$	$E, \Pi II'$	(2)
	Π	$E, -E$	(6)
Hextetrahedron	$-E$	$E, \Pi FF'$	(5)
	Π	$E, -FF'$	(17)
Gyroid	$-E$	$E, -FF'$	(17)
	Π	$E, \Pi FF'$	(5)
Hexoctahedron	either	$E, -\Pi II', \Pi FF', -FF'$	(19)
	either	$E, \Pi II', -\Pi FF', -FF'$	(18)
	either	$E, \Pi II', \Pi FF', FF'$	(15)

IX. Polyhedroidal Groups

56. The Polyhedroids. A regular polyhedroid is bounded by a certain number of regular polyhedrons, and may be considered as generated from an initial or basal polyhedron that has been distributed by a group of operators or transformations of symmetry. The group of transformations characterizing the symmetry of the regular polyhedroid will be the result of adding the distributing transformations to the group of collinear transformations that characterize the symmetry of the regular polyhedron. The axis of the collinear group is of course at right angles to the spread of three dimensions that contains the polyhedron. A planoid in this position will be manifolded into a regular poly-

hedroid without the aid of the transformations of symmetry belonging to the polyhedron, so that the framework of the polyhedroid is thus obtained, but not the full symmetry. In other words, so-called enantiomorphic forms are obtained. Our object is to visualize the groups of symmetry already obtained, and to do this we propose to examine the result of distributing the groups and sub-groups of the regular polyhedrons, identifying the result as an exemplification of a group of symmetry already obtained. All possible distributors are not considered. Each group will be described as a polyhedroid, with a qualifying prefix to show the symmetry assumed in the collinear group that is manifolded. It will also be seen that some duplication of groups occurs.

57. The Pentahedroid Distributor. The compound fivefold versor (Arts. 20*f*, 24*b*, 36) serves as distributor for the various pentahedroids, provided the axis of the collinear group is properly oriented. The groups to be distributed will be the group and sub-groups of the regular tetrahedron. If a center of symmetry be added to the regular pentahedroid, a figure is obtained bounded by ten equal solids each of which is a regular tetrahedron whose vertices have been cut off in such a way that the original faces have become regular hexagons. We shall speak of this figure as the dekahedroid, and the corresponding distributor is the negative of the compound fivefold versor. Adding a center of symmetry to any given pentahedroid will supply the corresponding dekahedroid.

58. The Octahedroid Distributor. The I, J, K group (Art. 3) manifolds any planoid not passing through the origin into a regular octahedroid, so no particular orientation is demanded. The group to be distributed is to be a sub-group of the group of symmetry of the cube. In other words the hexoctahedral group and its sub-groups.

59. The 16-hedroid Distributor. The regular 16-hedroid differs from the 8-hedroid in that solids and vertices are interchanged. The distributor will be the group E, II', JJ', KK',

II, $\Pi II'$, $\Pi JJ'$, $\Pi KK'$, and their negatives. The group to be distributed will be the hextetrahedron and its sub-groups, and the axis must lie in the direction of one of the vertices of the octahedroid that is the converter $C^{-1}(\ )C$ must be applied to the tetrahedral transformations as we have usually listed them.

60. The 24-hedroid Distributor. The tetartoid umbel (Art. 5) supplies the framework for a series of 24-hedroids, and no particular orientation is required for the axis of the collinear group to be distributed, which will in this case be the group and sub-groups of the hexoctahedron, the 24-hedroid being bounded by regular octahedrons.

61. The 48-hedroid Distributor. The gyroid umbel (Art. 6) supplies the framework for a figure bounded by 48 equal 14-hedrons, each obtained by removing the eight corners of a cube in such a way as to reduce the original faces to regular octagons. The groups to be distributed will be obtained from the hexoctahedron, and no particular orientation is required.

62. The 120-hedroid Distributor. The icosahedral umbel (Art. 7) produces the frame of the regular 120-hedroid, distributing the groups and sub-groups of the regular dodecahedron without any implication as to the location of the axis.

63. The 600-hedroid Distributor. If the plane WW' be added to the icosahedral umbel, and if the axis is properly chosen, the 600-hedroid is obtained. The axis must correspond to the vertex of the regular dodecahedron, so that the converter will be

$$L = \frac{1}{2}\left(E + A\,\frac{\sqrt{5}-1}{2}\right),$$

and the group to be distributed will be the group or a sub-group of the regular tetrahedron.

64. Basal Polyhedral Collinear Groups. Since the tetrahedron and cube have certain groups in common, it will be simpler to divide the discussion of these as is done in crystallography (Art. 51). All subgroups cannot always be used with a given distributor even when the complete group can be used. In the

case of basal rotation groups the 120-hedroid distributor gives difficulty, and in the case of perversion groups the perverters by demanding the presence of similar quaternions of the other class prevent the use of too complicated distributors with basal groups that are too simple. In the latter case, before accepting a distributor we must look to see if the planes AA', XX', and the like that would be formed in the course of the subsequent multiplication are already present, for otherwise they will be introduced, and we shall be distributing a basal group of a higher grade of symmetry than we intended. The groups and subgroups are supposed from solid geometry.

65. Subgroups of the Tetartoid. The tetartoid has three subgroups;

Monoclinic Sphenoid (Sphenoid)
KK'

Rhombic Bisphenoid (Rhombosphenoid)
II', JJ', KK'

Third Order Trigonal Hemi-pyramid (Trigonal)
AA' and its reciprocal.

The prefixes adopted to indicate these groups are added in parentheses after the crystallographic names. For the pentahedroid distributor, JJ' can be admitted to form a sphenoid pentahedroid, but in the other cases the symmetry of the basal group is increased, for

$$(WW'^3)^3KI'(WW'^3)^4 = AB'^5, \qquad (WW'^3)AB'^5(WW'^3)^4 = C^5D',$$

so that only the full tetartoid group can be used. In like manner in the case of the 600-hedroid distributor, we may admit a sphenoid form alone. The distributor involves a free gyroid umbel, and a free decimal versor of the other class. If the basal group contains only the Hamilton unit of the latter class peripheral to the given decimal versor, a whorl will be formed. If more is contained a free umbel of the second class is obtained,

and with the variety of transformations thus on hand, the collinear tetartoid group can be constructed in the required position. A table is given of the transformations of the regular tetrahedron in the four different positions in which the tetrahdron is being used.

66. Subgroups of the Hextetrahedron. Besides the tetartoid and its subgroups, the regular tetrahedron contains the following subgroups:

Class of the Monoclinic Dome. (Di)
$-P\Pi P$

Rhombic Hemipyramid (Tetrarhombic)
KK', $-P\Pi P$, $-Q\Pi Q$

Third Order Tetragonal Bisphenoid (Bisphenoid)
$\Pi Z^3Z'^7$ and its powers

Tetragonal Scalenohedron (Scalenohedral)
JJ', II', $\Pi Z^3Z'^7$ and its powers

Ditrigonal Hemipyramid (Ditrigonal)
AA', $-F\Pi F$, $-P\Pi P$, $-N\Pi N$

The prefixes are indicated. The octahedroid distributor will handle the scalenohedral basal group. The 16-hedroid distributor will handle them all. The umbel distributors raise the subgroups to the symmetry of the full group. The pentahedroid and dekahedroid distributors, as we have seen, will not serve for subgroups that contain other planes beside JJ'. If we convert WW'^3 by means of the simple perverters, planes can be released from the product. For example $C^5\Pi C$ converts WW'^3 into $A^2B'^4WW'^3$ and since WW'^3 is free we can multiply by its reciprocal and release the plane $A^2B'^4$. Hence the 5- and 10-hedroid distributors will only be able to use the bisphenoid in addition to the tetrahedron itself.

TRANSFORMATIONS OF THE BASAL TETRAHEDRAL GROUP IN THE VARIOUS POSITIONS IN WHICH IT IS TO BE ASSUMED FOR USE WITH VARIOUS DISTRIBUTORS

(Cf. Art. 57, 59, 63)

	(5-h)	(16-h)	(600-h)
Φ	$Y'\Phi Y'^{-1}$	$C^{-1}\Phi C$	$L^{-1}\Phi L$
AA'	AB'^{-1}	DA'	AA'
$A^5A'^5$	A^5B'	$D^5A'^5$	$A^5A'^5$
BB'	BC'^5	$A^5B'^5$	KW^3B'
$B^5B'^5$	B^5C'	AB'	$CW^3B'^5$
CC'	CD'^5	CC'	IW^3C'
$C^5C'^5$	C^5D'	$C^5C'^5$	$D^5W^7C'^5$
DD'	DA'^5	BD'	BW^7D'
$D^5D'^5$	D^5A'	$B^5D'^5$	$DW^3D'^5$
II'	$-IK'$	$-JI'$	B^5W^4I'
JJ'	JJ'	$-KJ'$	C^5W^8J'
KK'	KI'	IK'	JW^7K'
$-F\Pi F$	$C^5\Pi C$	$Z\Pi Z^7$	$(JW)^{-1}\Pi(JW)$
$-G\Pi G$	$A^5\Pi A$	$-Q\Pi Q$	$(KW)^{-1}\Pi(KW)$
$-M\Pi M$	$-I\Pi I$	$X\Pi X^7$	$(CW)^{-1}\Pi(CW)$
$-N\Pi N$	$-K\Pi K$	$-G\Pi G$	$(BW)^{-1}\Pi(BW)$
$-P\Pi P$	$D\Pi D^5$	$Y^7\Pi Y$	$(D^5W)^{-1}\Pi(D^5W)$
$-Q\Pi Q$	$B\Pi B^5$	$-N\Pi N$	$(A^5W)^{-1}\Pi(A^5W)$
$\Pi X^3X'^7$	$\Pi DB'^2$	$\Pi ZP'$	$\Pi B^2W^3W'^9$
$\Pi Y^3Y'^7$	$\Pi J'$	$\Pi FX'^7$	$\Pi A^4WI'W'^5$
$\Pi Z^3Z'^7$	$\Pi C^3A'$	$\Pi Y^7M'$	$\Pi C^3WD'W'^2$
$\Pi X^5X'$	$\Pi BD'^2$	$\Pi PZ'^3$	$\Pi WB'^2W'^3$
$\Pi Y^5Y'$	ΠJ^3	$\Pi X^5F'$	$\Pi DW^6K'W'^2$
$\Pi Z^5Z'$	$\Pi A^3C'$	$\Pi MY'^5$	$\Pi B^5WA'^5W'^4$

67. Subgroups of the Diploid. The symmetry of the diploid occurs as part of the symmetry of the octahedron or cube as well as part of the symmetry of the regular dodecahedron and icosahedron. For this reason the distributors that must be considered are the 8-, 24-, 48-, and 120-hedroid distributors. The sub-groups are:

Pedial Class (Not used)
$-\Pi$

Class of the Monoclinic Dome (Not used)
$-\Pi KK'$

Class of the Monoclinic Pyramid (Not used)
$-\Pi KK', KK', -\Pi$

Class of the Rhombic Hemipyramid (Not used)
$KK', -\Pi II', -\Pi JJ'$

Class of the Rhombic Bipyramid (Octorhombic)
$II', JJ', KK', -\Pi, -\Pi II', -\Pi JJ', -\Pi KK'$

Class of the Third Order Rhombohedron (Not used)
AA' and its reciprocal.

The diploid itself takes in addition to the transformations of the tetartoid the 12 perverters;

$\Pi AA', \Pi BB', \Pi CC', \Pi DD'$ and their reciprocals, $-\Pi II', -\Pi JJ', -\Pi KK', -\Pi$.

The class of the rhombic bipyramid will produce an octorhombic octahedroid, but owing to the shortage of planes, only the full diploid group can be used with the 24-hedroid distributor, and none of these basal groups can be used with the 48- and 120-hedroid distributors.

68. Subgroups of the Gyroid. The gyroid and its sub-groups must be considered for those distributors that handle hexahedral or octahedral symmetry, namely the 8-, 24-, and 48-hedroid distributors. There are three subgroups to be enumerated that did not occur in the tetartoid;

Class of the Third Order Tetragonal Hemipyramid
(Tetragonal)

ZZ' and its powers.

Class of the Tetragonal Trapezohedron
(Octo-Trapezohedral)

ZZ', etc., JJ', KK', II', PP', QQ'

Class of the Trigonal Trapezohedron
(Hexa-Trapezohedral)
AA', etc., FF', PP', NN'

These subgroups admit all three distributors.

69. Subgroups of the Hexoctahedron. The cube and regular octahedron embrace all the subgroups enumerated thus far, and in addition the following;

Class of the Third Order Tetragonal Bipyramid (Not used)
ZZ', etc., $-\Pi KK'$, $-\Pi$
Class of the Ditetragonal Hemipyramid (Not used)
ZZ', etc., $-\Pi II'$, $-\Pi JJ'$, $-\Pi PP'$, $-\Pi QQ'$
Class of the Ditetragonal Hemipyramid (Ditetragonal)
ZZ', etc., II', JJ', PP', QQ', $-\Pi II'$, $-\Pi JJ'$, $-\Pi PP'$, $-\Pi QQ'$
Class of the Trigonal Scalenohedron (Not used)
AA', etc., $-\Pi$, FF', PP', NN', $-\Pi FF'$, $-\Pi PP'$, $-\Pi NN'$

The ditetragonal bipyramid is ample enough to admit the octahedroid distributor, but otherwise only the full group can be used.

70. Subgroups of the Regular Icosahedron. Apart from the groups of the tetartoid and diploid, there remain to be enumerated the following subgroups of the regular dodecahedron, for which the only available distributor is the 120-hedroid distributor.

Third Order Pentagonal Hemipyramid (Pentagonal)
WW'' and its powers
Dipentagonal Hemipyramid (Not used)
WW'', etc., $-\Pi JJ'$, etc.
Pentagonal Rhombohedron (Not used)
WW'', etc., $-\Pi$
Pentagonal Trapezohedron (Dekatrapezohedral)
WW'', etc., FF', etc.
Pentagonal Scalenohedron (Not used)
WW'', etc., $-\Pi$, JJ', etc.

Gyro-icosahedral Group (Gyro-Dodecahedral)

The rotations of the regular icosahedron and dodecahedron.

The rotation groups presnt no difficulty, but the groups containing perversions will all be spoilt by the presence of planes derived from quaternions of the other class obtained by applying the perverters to convert the distributing umbel.

71. The Pentahedroids and Dekahedroids.

(*a*) Proto-Pentahedroid. This group consists of the compound versor WW'^2 and its powers alone, and is therefore a cyclic group. The proto-dekahedroid is the cyclic group produced by the negative of the same compound versor (Art. 57).

(*b*) Sphenoid-Pentahedroid. The result of applying the pentahedroid distributor (Art. 65) to the mono-clinic sphenoid is a blended whorl whose principal group is the cyclic group obtained from the compound versor WW'^2 attended by laterals derived from JJ', but without medials (Art. 26). The corresponding dekahedroid adds a center.

(*c*) Bisphenoid-Pentahedroid. Distributing the third order tetragronal bisphenoid (Art. 66) in the same way is equivalent to adding perverted peripherals $\Pi C^2A'$ to the group last described (Art. 36). The corresponding dekahedroid merely adds a center.

(*d*) Tetartoid-Pentahedroid. This is the rotation group of the regular pentahedroid (Art. 49, No. 32), and consists of two icosahedral umbels blending in compound symmetry. The addition of a center gives the corresponding dekahedroid (Art. 49, No. 35).

(*e*) Tetrahedral-Pentahedroid. Distributing the group of the regular tetrahedron, Art. 51, we obtain the group of the regular pentahedroid (Art. 49, No. 33). A corresponding dekahedroid (Art. 49, No. 38) is obtained by the addition of a center.

(*f*) Alternating-Pentahedroid. Group number (34) of Art. 49 has not been described, but is related to group number (33) in much the same way that the diploid is related to the tetrahedron, and therefore may be considered an alternating form of the latter.

72. The Octahedroids.

(*a*) Proto-Octahedroid. The quaternion versor group consisting of the three spheniodals and their powers develops any planoid not passing through the origin into a regular octahedroid (Art. 3 and 58).

(*b*) Sphenoid-Octahedroid. Distributing the monoclinic sphenoid (Art. 65) with the octahedroid distributor gives a group that can be analyzed into the combination, fully resolved, of a fourth order quaternion versor with a whorl having a stem of the fourth order (Art. 29*c*).

(*c*) Rhombosphenoid-Octahedroid. The rhombic bisphenoid (Art. 65) distributed by the octahedroid distributor consists of two complete whorls with stems of the fourth order, fully resolved (Art. 30*e*).

(*d*) Octorhombic and Scalenohedral Octahedroids. If the rhombic bipyramid (Art. 67) and the tetragonal scalenohedron (Art. 66) are each distributed by the octahedroid distributor, the groups obtained are similar as they can be converted into each other by the converter $Z^1(\ )Z$. The same group results from adding perverters to the resolved whorls last mentioned (Art. 37*d*).

(*e*) Tetragonal-Octahedroid. The distribution of the third order tetragonal hemipyramid (Art. 68) gives the same result as blending an eightfold quaternion versor with a whorl with stem of the same order, the stem being semi-resolved and having alternate peripherals free (Art. 29*c*).

(*f*) Octotrapezohedral-octahedroid. The distribution of the tetragonal trapezohedron (Art. 68) by the octahedroid distributor gives us a rotation group formed from two whorls with stems of the eighth order. The stems are blended, semi-resolved, and alternate peripherals are free. In other words, we have the group with medials and laterals and semi-resolved principal (Art. 30*d*).

(*g*) Ditetragonal Octahedroid. The distribution of the ditetragonal bipyramid (Art. 69) represents the addition of primary perverters to the preceding group (Art. 37*c*).

(*h*) Trigonal Octahedroid. The third order trigonal hemipyramid (Art. 65) is expanded into a rotation group that results from blending all the tetartoidals of a tetartoid umbel with a sixth order quaternion versor of the other class, and allowing the sphenoidals to remain free (Art. 39*a*).

(*i*) Hexatrapezohedral Octahedroid. The rotation group obtained from the trigonal trapezohedron (Art. 68) by applying the octahedroid distributor consists of a whorl whose stem is of the sixth order that has been blended with a gyroid umbel in such a way that the stem of the whorl blends the tetartoidals, while the peripherals of the whorl take up the cardinals and gyroidals, and the sphenoidals of the umbel remain free (Art. 40*b*).

(*j*) Tetartoid-Octahedroid. If the distributed group is the tetartoid of solid geometry (Art. 51), i.e. the collinear tetartoid (1) of Article 49, we then obtain number (9) of the same article, the tetartoid with released sphenoidals, a partial blending of two tetartoid umbels.

(*k*) Tetrahedral-Octahedroid. Taking the hextetrahedral group (Art. 51) with the octahedroid distributor gives us number (11) of article 49, a perversion of the preceding group.

(*l*) Diploid-Octahedroid. Starting with the diploid (Art. 51) we obtain another perversion of the same rotation group, number (10) of article 49.

(*m*) Gyroid-Octahedroid. Distributing the gyroid basal group, (Art. 51) we obtain number (22) of article 49 a partial blending of two gyroid umbels with released sphenoidals. This is the rotation group of the regular octahedroid.

(*n*) Octahedral-Octahedroid. The distribution of the full hexoctahedron (Art. 51) gives us the complete symmetry of the regular octahedroid, the result of adding perverters to the preceding group, (23) of article 49.

73. The Hexadekahedroids.

(*a*) Proto-16-hedroid. The 16-hedroid distributor consists of two whorls, stems of the fourth order, principal group a plane of twofold symmetry, with laterals and without medials, with primary simple perverters (Art. 35, 59).

(*b*) Di-16-hedroid. The class of the monoclinic dome (Art. 66) expands into a group formed from two whorls with stems of the eighth order, forming a plane of fourfold symmetry with laterals, without medials, with primary simple perverters (Art. 35).

(*c*) Sphenoid-16-hedroid. The monoclinic bisphenoid (Art. 65) expands into a group derived from two fourth order whorls, principal resolved, with laterals, without medials, with primary simple perverters (Art. 35).

(*d*) Tetrarhombic-16-hedroid. The rhombic hemi-pyramid (Art. 65) expands into a group derived from two whorls with stems of the 8th order, principal semi-resolved, with laterals, without medials, with primary simple perverters (Art. 35).

(*e*) Bisphenoid-16-hedroid. The third order tetragonal bisphenoid (Art. 66) expands into a group derived from two whorls with stems of the 8th order, with medials and laterals and " reduced principal " with primary perversion (Art. 37*b*).

(*f*) Rhombosphenoid-16-hedroid. The rhombic bisphenoid (Art. 65) is expanded into a group derived from two whorls, totally unblended, stems of the fourth order, perverted (Art. 37*d*).

(*g*) Scalenohedral-16-hedroid. The tetragonal scalenohedron (Art. 66) is expanded into a group derived from two whorls with stems of the 8th order, with medials and laterals, semi-resolved principal and primary perversion (Art. 37*c*).

(*h*) Trigonal-16-hedroid. The third order trigonal hemipyramid (Art. 65) is distributed into a group derived from two completely blended tetartoid umbels with diploid perverters and a center (7) of article 49.

(*i*) Ditrigonal-16-hedroid. The ditrigonal hemipyramid (Art. 65) expands into two completely blended gyroid umbels, with a center and perversion (21).

(*j*) Tetartoid-16-hedroid. By applying the 16-hedroid distributor to the collinear tetartoid (Art. 51) we obtain the same group as by applying the octahedroidal distributor to the col-

linear diploid (Art. 72*l*), a group formed from the blending of two tetartoid umbels with released sphenoidals and diploid perversion (10) of article 49.

(*k*) Tetrahedral-16-hedroid. The distribution of the hextetrahedral basal group (Art. 51) by the 16-hedroid distributor is likewise equivalent to the distribution of the hexoctahedral basal group by the octahedroid distributor (Art. 72*n*), and describes the symmetry of the regular 8- and 16-hedroids, (23) of article 49.

74. The 24-hedroids.

(*a*) Proto-24-hedroid. The 24-hedroid distributor is the tetartoid umbel (Art. 5, 60).

(*b*) Sphenoid-24-hedroid. Distributing the monoclinic sphenoid (Art. 65) we obtain the effect of combining a Hamilton Unit with a tetartoid umbel of the other class without blending (Art. 39).

(*c*) Rhombosphenoid-24-hedroid The rhombic bisphenoid (Art. 65) expands into the unblended combination of a whorl with stem of the fourth order with a tetartoid umbel (Art. 39).

(*d*) Tetragonal-24-hedroid. By a distribution of the third order tetragonal hemipyramid (Art. 68) there is obtained the unblended combination of an 8th order quaternion versor with a tetartoid umbel (Art. 39).

(*e*) Octotrapezohedral-24-hedroid. The tetragonal trazpeohedron (Art. 68) is developed into a group formed from the blending of a whorl with an eightfold stem with a gyroid umbel, alternate peripherals being free and the stem being semi-released (Art. 41*a*).

(*f*) Trigonal-24-hedroid. The distribution of the third order trigonal hemipyramid (Art. 65) gives rise to the unblended combination of a quaternion versor of the sixth order with the tetartoid umbel (Art. 39).

(*g*) Hexatrapezohedral-24-hedroid. The trigonal trapezohedron (Art. 68) is expanded into a group consisting of a whorl with stem of the sixth order blended with a gyroid umbel in such a way that all the peripherals blend with each cardinal and each

gyroidal, the stem and the tetartoid umbel remaining free (Art. 41*b*).

(*h*) Tetartoid-24-hedroid. The distribution of the collinear tetartoid basal group (Art. 51) results in a pair of unblended tetartoid umbels (12) of article 49.

(*i*) Tetrahedral-24-hedroid. Using the hextetrahedral base (Art. 51) instead of the tetartoid adds perverters to the preceding group giving number (13) of article 49.

(*j*) Diploid-24-hedroid. The diploid basal group (Art. 51) gives rise to the preceding group merely turned through an angle.

(*k*) Gyroid-24-hedroid. The distribution of the gyroid basal group (Art. 51) yields the rotation group of the regular 24-hedroid and consists of two blended gyroid umbels with released tetartoidals (24) of article 49.

(*l*) Octahedral-24-hedroids. The distribution of the hexoctahedral group (Art. 51) gives the symmetry of the regular 24-hedroid (25) of article 49.

(*m*) Alternating 24-hedroid. The addition of the compound perverter ΠF to the gyroid-24-hedroid produces a group (26) of article 49 that has not been developed by the distribution of any of our collinear groups. It may be convenient to call this group an alternating 24-hedroid.

75. The 48-hedroids.

(*a*) Proto-48-hedroid. The distributor is the gyroid umbel (Arts. 6 and 61). All the rotation groups that follow are completely resolved.

(*b*) Sphenoid-48-hedroid. Distributing the class of the monoclinic sphenoid (Art. 65) is the same as combining a fourfold quaternion versor with a gyroid umbel without blending (Art. 39).

(*c*) Rhombosphenoid-48-hedroid. The rhombic bisphenoid (Art. 65) is expanded into a group obtained by combining with a gyroid umbel a whorl with stem of the fourth order (Art. 40).

(*d*) Tetragonal-48-hedroid. The third order tetragonal hemipyramid (Art. 68) is developed into an eightfold versor combined without blending with a gyroid umbel (Art. 39).

(*e*) Octotrapezohedral-24-hedroid. The distribution of the tetragonal trapezohedron (Art. 68) combines a whorl with stem of the 8th order with a gyroid umbel without blending (Art 40).

(*f*) Trigonal-48-hedroid. The third order trigonal hemipyramid (Art. 65) is developed into the unblended combination of a sixth order versor with a gyroid umbel (Art. 39).

(*g*) Hexatrapezohedral-48-hedroid. The trigonal trapezohedron (Art. 68) becomes the unblended combination of a whorl with stem of the sixth order with a gyroid umbel (Art. 40).

(*h*) Tetartoid-48-hedroid. The distribution of the collinear tetartoid basal group (Art. 51) means the combination of a tetartoid umbel with a gyroid umbel. Unlike umbels cannot be blended (Art. 43).

(*i*) Gyroid-48-hedroid. The distribution of the gyroid basal group (Art. 51) is the same as the combination of two gyroid umbels without blending (27) of article 49.

(*j*) Octahedral-48-hedroid. A distribution of the hexoctahedron (Art. 51) is a perversion of the preceding group (28) of article 49.

76. The 120-hedroids.

(*a*) Proto-120-hedroid. The distributor is the icosahedral umbel (Arts. 7, 62).

(*b*) Sphenoid-120-hedroid. From the monoclinic sphenoid (Art. 65), a fourfold versor and the icosahedral umbel (Art. 39).

(*c*) Rhombosphenoid-120-hedroid. From the rhombic bisphenoid (Art. 65), a whorl with stem of the fourth order with the icosahedral umbel (Art. 40).

(*d*) Trigonal-120-hedroid. The third order trigonal hemipyramid (Art. 65) gives the combination of a sixth order versor with the icosahedral umbel (Art. 39).

(*e*) Pentagonal-120-hedroid. The third order pentagonal hemipyramid (Art. 70) produces the combination of a tenth order versor with the icosahedral umbel (Art. 39).

(*f*) Hexatrapezohedral-120-hedroid. The trigional trapezohedron (Art. 68) expands into the combination of a whorl with stem of the sixth order with the same umbel (Art. 40).

(*g*) Dekatrapezohedral-120-hedroid. The pentagonal trapezohedron (Art. 70) gives a whorl with stem of the tenth order combined with the same order (Art. 40).

(*h*) Tetartoid-120-hedroid. Distributing the collinear tetartoid (Art. 51) is the same as combining a tetartoid umbel with a gyroid umbel (Art. 43).

(*i*) Gyrododecahedral-120-hedroid. The distribution of the rotation group of the regular dodecahedron (Art. 51) gives the unblended combination of two icosahedral umbels (39) of article 49.

(*j*) Dodecahedral-120-hedroid. The distribution of the group of symmetry of the regular dodecahedron (Art. 51) is a perversion of the preceding group (40) of article 49.

77. The 600-hedroids.

(*a*) Proto-600-hedroid. This group, which consists of the distributor (Art. 63) alone, consists of a tenth order quaternion versor combined with an icosahedral umbel of the other class (Art. 39). This is, of course, the same group as the pentagonal-120-hedroid (Art. 76*e*).

(*b*) Sphenoid-600-hedroid. Distributing the monoclinic sphenoid (Art. 65) with the 600-hedroid distributor gives the same group as the dekatrapezohedral-120-hedroid (Art. 79*g*), a whorl of the tenth order combined with the icosahedral umbel (Art. 40).

(*c*) Tetartoid-600-hedroid. The distribution of the collinear tetartoid (Art. 51) repeats the gyrododecahedral-120-hedroid (Art. 76*i*), giving the rotation group of the regular 120- and 600-hedroids (39) of article 49.

(*d*) Tetrahedral-600-hedroid. The distribution of the hextetrahedral (Art. 51) collinear group gives the full group of symmetry of the regular 600-hedroid, (40) of article 49, a perversion of the preceding group. It is also the symmetry of the regular 120-hedroid, and therefore the same as the dodecahedral-120-hedroid (Art. 76*j*).

MULTIPLICATION TABLE FOR THE GYROID UMBEL

E	A	A^5	B	B^5	C	C^5	D	D^5	I	J	K	X	X^7	Y	Y^7	Z	Z^7	F	G	M	N	P	Q
A	A^2	E	D^5	I^3	B^5	K^2	C^5	J^3	C	B	D	Y^7	M^2	Z^7	Q^3	X^7	G^3	N^3	Y	Z	P^3	F^3	X
A^5	E	A^4	J	C	I	D	K	B	B^2	D^2	C^3	Q	Z	G	X	M	Y	P^3	Z^3	X^3	F^3	N^3	X^3
B	C^5	I	B^2	E	D^5	J	A^5	K^3	D^4	A^4	C	N^3	Y	Q	Z^7	X	F	Y^3	M	P	Z	G^3	X^3
B^5	J^3	D	E	B^4	K	A	I^3	C	A^5	C^5	D^2	Z	Q^3	X^7	F^3	N	Y^7	Z^7	P^3	G	X^5	M	Y
C	D^5	K	A^5	J^3	C^2	E	B^5	I	A^4	D	B^4	M	Y^7	Z	P	F^3	X	Y	N	Z^3	Q^3	X^3	G
C^5	I^3	B	K^2	D	E	C^4	J	A	D^5	B^2	A^5	Z^7	P^3	F	X^7	Y	M^3	Z^5	Q	X	G	Y^7	N^3
D	B^5	J	C^5	K	A^5	I^3	D^2	E	B	C^4	A^4	Y	N	P^3	Z	G	X^7	M^3	Y^3	Q	Z^3	X	F
D^5	K^2	C	I	A	J^3	B	E	D^4	C^2	A^5	B^5	P	Z^7	X	G^3	Y^7	N^3	Q	Z	F^3	X^7	Y^5	M
I	B	C^2	A^4	D^5	D^4	A^5	C	B^5	$-E$	K	J^2	X^3	X	M	N^3	P	Q	G	F^3	Y^3	Y^7	Z^5	Z^3
J	D	B^2	C^4	A^5	B	D^5	A^4	C^5	K^2	$-E$	I	F	G	Y^3	Y	Q	P^3	X^5	X^3	N^3	M	Z^7	Z^5
K	C	D^2	D	C^5	A^4	B^5	B^4	A^5	J	$-J^2$	$-E$	G	F^3	N	M	Z^3	Z	X^7	X^5	Y^3	Y^3	Q	P^3
X	Z^7	M	Q	Y^7	P	Y	Z	N^3	X^3	G	F^3	I	E	A^5	D^5	C	B	J	K	C^3	B^5	D^4	A^4
X^7	Q^3	Y	Z^7	N	Z	M^3	P^3	Y^3	X	F	G	E	I^3	C^5	B^5	D	A	K^2	J	A^5	D^2	C	B
Y	X^7	Q	F	Z	X	P^3	G	Z^7	N^3	Y^3	M	B	D	J	E	A^5	C^5	C^4	A^4	I	K	D^5	B^2
Y^7	G^3	Z	X	Q^3	F^3	Z^7	X^7	P	M	Y	N	C	A	E	J^3	B^5	D^5	B	D	K	I^3	C^2	A^5
Z	Y^7	G	Y	F^3	M	X^7	N	X	Q	P^3	Z^3	A^5	B^5	D	C	K	E	C^5	D^2	A^4	B^4	I	J
Z^7	M^3	X	N^3	X^7	Y^7	F	Y	G^3	P	Q	Z	D^5	C^5	B	A	E	K^2	B^2	A^5	C	D	J^3	I
F	P^3	N^3	Z^5	Y	Z^7	Y^3	Q	M^3	G^3	X^3	X	K^3	J	B^2	C^5	B	C^4	$-E$	I	D^5	A^5	A	D^4
G	Z	Y^3	P^3	M	Q	N	Z^3	Y	F	X^5	X^3	J	K	D^2	A^5	A^4	D	I^3	$-E$	B^2	C^2	B	C^4
M	X	Z^3	G	P	X^3	Z	F^3	Q	Y^3	N	Y^5	A^4	C	K	I	C^3	A^5	D	B^4	$-E$	J^3	B^3	D^2
N	F^3	P^3	X^7	Z^3	G	Q^3	X^5	Z	Y	M^3	Y^3	D	B^4	I^3	K	D^2	B^5	A	C^4	J	$-E$	A^5	C^5
P	N^3	F^3	M	G^3	Y^5	X	Y^7	X^3	Z^3	Z	Q^3	C^3	D^5	C	D^4	J^3	I	A^5	B^5	B^4	A	$-E$	K
Q	Y	X^3	Y^3	X	N^3	G	M	F	Z^5	Z^3	P	B^2	A^5	A^4	B	I	J	D^2	C^3	D^4	C	K^2	$-E$

The prefactors are given at the left, the postfactors at the top.

Conversion Table for the Gyroid Umbel

Values of $R^{-1}SR$

S/R	A	A^5	B	B^5	C	C^5	D	D^5	I	J	K	X	X^7	Y	Y^7	Z	Z^7	F	G	M	N	P	Q
A	A	A	C	D	D	B	B	C	D	C	B	B^5	C^5	D^5	B^5	C^5	D^5	A^5	D^5	C^5	A^5	A^5	B^5
B	D	C	B	B	A	D	C	A	C	D	A	D^5	A^5	A^5	C^5	D^5	C^5	C^5	B^5	B^5	D^5	B^5	A^5
C	B	D	D	A	C	C	A	B	B	A	D	A^5	D^5	B^5	D^5	B^5	A^5	B^5	C^5	A^5	C^5	D^5	C^5
D	C	B	A	C	B	A	D	D	A	B	C	C^5	B^5	C^5	A^5	A^5	B^5	D^5	A^5	D^5	B^5	C^5	D^5
I	J	K	J	K^3	J^3	K	J^3	K^3	I	I^3	I^3	I	I	K	K^3	J^3	J	I^3	I^3	K	K^3	J^3	J
J	K	I	K^3	I	K^3	I^3	K	I^3	J^3	J	J^3	K^3	K	J	J	I	I^3	K^3	K	J^3	J^3	I^3	I
K	I	J	I^3	J^3	I	J^3	I^3	J	K^3	K^3	K	J	J^3	I^3	I	K	K	J^3	J	I	I^3	K^3	K^3
X	Y	Z	Y	Z^7	Y^7	Z	Y^7	Z^7	X	X^7	X^7	X	X	Z	Z^7	Y^7	Y	X^7	X^7	Z	Z^7	Y^7	Y
Y	Z	X	Z^7	X	Z^7	X^7	Z	X^7	Y^7	Y	Y^7	Z^7	Z	Y	Y	X	X^7	Z^7	Z	Y^7	Y^7	X^7	X
Z	X	Y	X^7	Y^7	X	Y^7	X^7	Y	Z^7	Z^7	Z	Y	Y^7	X^7	X	Z	Z	Y^7	Y	X	X^7	Z^7	Z^7
F	N	P	N^3	Q	M^3	P^3	M	Q^3	F^3	G	G^3	G^3	G	Q	P^3	N^3	M^3	F	F^3	Q^3	P	N	M
G	M	Q	M^3	P	N^3	Q^3	N	P^3	G^3	F	F^3	F	F^3	P^3	Q	M	N	G^3	G	P	Q^3	M^3	N^3
M	Q	G	P^3	G^3	P	F^3	Q^3	F	N^3	M^3	N	Q	P	N	N^3	F^3	G	Q^3	P^3	M	M^3	G^3	F
N	P	F	Q^3	F^3	Q	G^3	P^3	G	M^3	N^3	M	P^3	Q^3	M^3	M	G	F^3	P	Q	N^3	N	F	G^3
P	F	N	G	M^3	F^3	M^3	G^3	N^3	Q	Q^3	P^3	M	N^3	F^3	G^3	Q^3	Q	N	M^3	G	F	P	P^3
Q	G	M	F	N^3	G^3	N	F^3	M^3	P	P^3	Q^3	N^3	M	G	F	P	P^3	M^3	N	F^3	G^3	Q^3	Q

The Icosahedral Umbel

R	RW	RW^2	RW^3	RW^4
E	$\{E\ I\ K\}$	$\{I\ -K\ -E\}$	$\{I\ -K\ E\}$	$\{-E\ I\ K\}$
A	$\{-J\ E\ K\}$	$\{-J\ I\ -E\}$	$\{I\ -J\ -K\}$	$\{I\ K\ E\}$
A^5	$\{J\ I\ -E\}$	$\{J\ -K\ -I\}$	$\{-K\ J\ E\}$	$\{-K\ -E\ I\}$
B	$\{-K\ I\ -J\}$	$\{-K\ -E\ -I\}$	$\{-E\ -K\ J\}$	$\{-E\ -J\ I\}$
B^5	$\{E\ K\ J\}$	$\{E\ I\ -K\}$	$\{I\ E\ -J\}$	$\{I\ J\ K\}$
C	$\{I\ K\ -E\}$	$\{I\ J\ -K\}$	$\{J\ I\ E\}$	$\{J\ -E\ K\}$
C^5	$\{-K\ E\ I\}$	$\{-K\ -J\ -E\}$	$\{-J\ -K\ -I\}$	$\{-J\ I\ E\}$
D	$\{E\ J\ I\}$	$\{E\ -K\ -J\}$	$\{-K\ E\ -I\}$	$\{-K\ I\ J\}$
D^5	$\{I\ -J\ K\}$	$\{I\ -E\ J\}$	$\{-E\ I\ -K\}$	$\{-E\ K\ -J\}$
I	$\{I\ -E\ -J\}$	$\{-E\ J\ -I\}$	$\{-E\ J\ I\}$	$\{-I\ -E\ -J\}$
J	$\{J\ -K\ I\}$	$\{-K\ -I\ -J\}$	$\{-K\ -I\ J\}$	$\{-J\ -K\ I\}$
K	$\{K\ J\ -E\}$	$\{J\ E\ -K\}$	$\{J\ E\ K\}$	$\{-K\ J\ -E\}$

Table for Changing Prefactors into Postfactors

R	WR	W^2R	W^3R	W^4R	R
A	C^5W	KW^4	A^5W^2	CW^3	A
A^5	CW	AW^3	C^5W^4	K^2W^2	A^5
B	B^5W^4	IW	D^5W^3	D^4W^2	B
B^5	I^2W^2	D^2W^4	DW^3	BW	B^5
C	AW^2	C^5W^3	K^2W	A^5W^4	C
C^5	KW^3	A^5W	CW^2	AW^4	C^5
D	B^4W^3	B^5W^2	I^2W^4	D^5W	D
D^5	DW^4	BW^2	B^2W	IW^3	D^5
I	D^5W^2	D^4W	BW^4	B^2W^3	I
J	J^3W^4	J^3W^3	J^3W^2	J^3W	J
K	A^2W^3	C^4W^4	AW	C^5W^2	K

Conversions of W^6

R	$R^{-1}WR$	$R^{-1}W^2R$	$R^{-1}W^3R$	$R^{-1}W^4R$
A	DW	C^2W^4	A^4W^2	IW^3
A^5	B^5W	A^2W^3	K^3W^4	D^4W^2
B	B^4W^4	A^5W	CW^3	IW^2
B^5	DW^2	KW^4	A^5W^3	B^2W
C	I^3W^2	C^4W^3	A^2W	BW^4
C^5	B^4W^3	KW	C^2W^2	D^5W^4
D	I^2W^3	AW^3	C^5W^4	D^4W
D^5	D^2W^4	C^5W^2	K^2W	BW^3
I	B^5W^2	CW	AW^4	D^5W^2
J	W^5	W^3	W^7	W^6
K	D^2W^2	A^4W^4	C^4W	B^2W^2

Conversions by W^6

R	W^5RW	W^3RW^2	W^7RW^3	W^5RW^4
A	C^4W^4	A^3W^4	KW^2	C^5
A^5	KW^3	C^5W	AW	C
B	DW^3	D^5	I^2W^4	B^5W^2
B^5	B^4W^2	D	D^2W^2	I^2W
C	A^5	KW^3	C^5W	AW
C^5	A	C^4W^4	A^2W^4	KW^2
D	D^4W^2	I^2W	B^5	B^4W^2
D^5	I^2W^4	B^5W^3	B	DW^3
I	B^5W^4	BW	DW^4	D^5W
J	JW^2	JW^4	J^2W	J^2W^3
K	C^2W^2	A^4W^2	C^4W^2	A^2W^4

A NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE OF A SEQUENCE OF LAGRANGE POLYNOMIALS [1]

By George Rutledge

1. Introduction. The Lagrange polynomial which takes the values of arbitrary $f(x)$ for integral values of x, from $x = -n$ to $x = +n$ including zero, will be denoted by $P^{(2n)}(x)$. It may be written [2]

$$x\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)\left\{\frac{f(0)}{x}+\sum_{j=1}^{n}\frac{\binom{2n}{n-j}}{\binom{2n}{n}}(-1)^j\left(\frac{f(-j)}{x+j}+\frac{f(j)}{x-j}\right)\right\}. \quad (1)$$

If we write $f(x)$ in the form

$$f(0)+\frac{f(x)-f(-x)}{2}+\frac{f(x)+f(-x)-2f(0)}{2} = f(0)+G(x)+F(x), \quad (2)$$

then $P^{(2n)}(x)$ may be written

$$f(0)+\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)\cdot x\sum_{j=1}^{n}\frac{\binom{2n}{n-j}}{\binom{2n}{n}}(-1)^j\left(\frac{f(-j)-f(0)}{x+j}+\frac{f(j)-f(0)}{x-j}\right)$$

$$=f(0)+\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)\cdot\sum_{j=1}^{n}\frac{\binom{2n}{n-j}}{\binom{2n}{n}}(-1)^{j-1}\frac{j^2}{j^2-x^2}\frac{2G(j)}{j}x$$

[1] Presented to the *American Mathematical Society*, Feb. 21, 1930.

[2] For derivation of this form from the classical formula of Lagrange see this *Journal*, vol. 8 (1929), p. 14.

$$+\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2}\right)\cdot\sum_{j=1}^{n}\frac{\binom{2n}{n-j}}{\binom{2n}{n}}(-1)^{j-1}\frac{j^2}{j^2-x^2}\frac{2F(j)}{j^2}x^2. \quad (3)$$

Thus $P^{(2n)}(x)$ is expressed in the form $f(0) + P_1^{(2n-1)}(x) + P_2^{(2n)}(x)$, where $P_1^{(2n-1)}(x)$ and $P_2^{(2n)}(x)$ are the odd and even polynomials determined, respectively, by the odd and even functions $G(x)$ and $F(x)$. We note that $G(0) = F(0) = 0$.

The coefficients of x and x^2 in (3) are now obtained at once in the form [3]

$$\begin{aligned} a_1^{(n)} &= \sum_{j=1}^{n}\frac{\binom{2n}{n-j}}{\binom{2n}{n}}(-1)^{j-1}\frac{2G(j)}{j}, \\ a_2^{(n)} &= \sum_{j=1}^{n}\frac{\binom{2n}{n-j}}{\binom{2n}{n}}(-1)^{j-1}\frac{2F(j)}{j^2}. \end{aligned} \quad (4)$$

When the coefficients (4) are convergent as n becomes infinite we shall write

$$\begin{aligned} \lim_{n\to\infty} a_1^{(n)} &= (VP)\sum(-1)^{j-1}\frac{2G(j)}{j} = a_1, \\ \lim_{n\to\infty} a_2^{(n)} &= (VP)\sum(-1)^{j-1}\frac{2F(j)}{j^2} = a_2, \end{aligned} \quad (5)$$

since convergence of the expressions (4) defines summability of the series in (5) by the method of de la Vallée-Poussin.[4]

Similarly, the limit of (3), when it exists, is

$$f(0) + \frac{\sin \pi x}{\pi x}\cdot(VP)\sum(-1)^{j-1}\frac{j^2}{j^2-x^2}\frac{2G(j)}{j}x + \frac{\sin \pi x}{\pi x}\cdot(VP)\sum(-1)^{j-1}\frac{j^2}{j^2-x^2}\frac{2F(j)}{j^2}x^2. \quad (6)$$

[3] For various forms of these coefficients see the following papers by the author: *Transactions of the American Mathematical Society*, vol. 26 (1924), pp. 113–123, and vol. 31 (1929), pp. 807–820.

[4] *Bull. de l'Acad. Roy. de Belgique* (*Classe des Sciences*), 1908, pp. 193–254; also *Rice Institute Pamphlet*, vol. XII (1925), p. 117.

In this paper we shall prove that the series in (6) are summable (VP), for non-integral x, which is not zero, when and only when the series in (5) are summable (VP). We shall establish this

Theorem. If $a_1^{(n)} \to a_1$, and $a_2^{(n)} \to a_2$, as n becomes infinite, then $P^{(2n)}(x)$ converges uniformly in any finite region of the x-plane to the integral function

$$\begin{aligned}\Phi(x) = f(0) &+ a_1x + a_2x^2 \\ &- \sum_{j=1}^{\infty} \alpha_{1j}(a_1 - a_1^{(j)})x^3 - \sum_{j=1}^{\infty} \alpha_{1j}(a_2 - a_2^{(j)})x^4 \\ &+ \sum_{j=2}^{\infty} \alpha_{2j}(a_1 - a_1^{(j)})x^5 + \sum_{j=2}^{\infty} \alpha_{2j}(a_2 - a_2^{(j)})x^6 \\ &- \sum_{j=3}^{\infty} \alpha_{3j}(a_1 - a_1^{(j)})x^7 - \sum_{j=3}^{\infty} \alpha_{3j}(a_2 - a_2^{(j)})x^8 \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ,\end{aligned}$$

where [5]

$$\alpha_{ij} = \frac{1}{j^2} \sum_{1}^{j-1} \frac{1}{r_1^2 r_2^2 \cdots r_{i-1}^2}, \qquad \sum_{j=i}^{\infty} \alpha_{ij} = \frac{\pi^{2i}}{(2i+1)!}.$$

Conversely, convergence of $P^{(2n)}(x)$, for a value of x which is not a positive or negative integer or zero, can occur in no other way than through existence of the limits $a_1^{(n)} \to a_1$, $a_2^{(n)} \to a_2$.

2. Application of an Extension of Toeplitz' Theorem to (VP) Summation. It is apparent that existence of the limit of $P^{(2n)}(x)$ will follow from existence of the limits $a_1^{(n)} \to a_1$, $a_2^{(n)} \to a_2$ if existence of the first of the limits

$$(VP) \sum u_j, \qquad (VP) \sum \frac{j^2}{j^2 - x^2} u_j \tag{7}$$

always implies existence of the second (x not an integer).

[5] The expression $\frac{1}{j^2} \sum_{1}^{j-1} \frac{1}{r_1^2 r_2^2 \cdots r_{i-1}^2}$ indicates the sum of all those products of the squared reciprocals of the first j integers, taken i at a time, which involve j. Then $\sum_{j=1}^{\infty} \alpha_{ij}$ is a particular arrangement of the products i at a time of the squared reciprocals of all positive integers. This sum of products is well known through its connection with

$$\frac{\sinh \pi x}{\pi x} = \prod_{1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = 1 + \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \alpha_{ij} x^{2i}.$$

In order to prove that this dependence of the second of these limits on the first is a property of the de la Vallée-Poussin process of summation, and also to derive the form of the limit of $P^{(2n)}(x)$ we apply a general theorem due to Kojima.[6] The theorem is as follows.

A necessary and sufficient condition that

$$\lim_{n\to\infty} (c_{n1}s_1 + c_{n2}s_2 + \cdots + c_{nn}s_n)$$

shall exist and be finite in all cases in which $\lim_{n\to\infty} s_n$ *exists and is finite is that*

(1) $\lim_{n\to\infty} c_{nk} = c_k$;

(2) $\lim_{n\to\infty} (c_{n1} + c_{n2} + \cdots + c_{nn}) = A$;

(3) $|c_{n1}| + |c_{n2}| + \cdots + |c_{nn}| < M$ *for every* n, *where* M *is a constant independent of* n.

When these conditions are satisfied,

$$\lim_{n\to\infty} (c_{n1}s_1 + c_{n2}s_2 + \cdots + c_{nn}s_n) = A \cdot \lim_{n\to\infty} s_n + \sum_{k=1}^{\infty} c_k(s_k - \lim_{n\to\infty} s_n),$$

the latter series being always absolutely convergent.

Let us place

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \quad s_3 = u_1 + u_2 + u_3, \quad \cdots, \tag{8}$$

and

$$\mu_{nj} = \frac{\binom{2n}{n-j}}{\binom{2n}{n}}, \qquad \nu_j = \frac{j^2}{j^2 - x^2}. \tag{9}$$

Then in order to prove that existence of the first of the limits (7) implies existence of the second we must prove that con-

[6] *Tôhoku Mathematical Journal*, vol. 12 (1917), pp. 291–326. See Carmichael, *Bulletin of the American Mathematical Society*, vol. 25 (1918), p. 110; J. Schur, *Journal für Mathematik*, vol. 151 (1921), p. 82.

vergence of the sequence

$$s_n' = (\mu_{n1} - \mu_{n2})s_1 + (\mu_{n2} - \mu_{n3})s_2 + \cdots + (\mu_{nn-1} - \mu_{nn})s_{n-1} + \mu_{nn}s_n \quad (10)$$

implies convergence of the sequence

$$s_n'' = (\nu_1\mu_{n1} - \nu_2\mu_{n2})s_1 + (\nu_2\mu_{n2} - \nu_3\mu_{n3})s_2 + \cdots + \nu_n\mu_{nn}s_n. \quad (11)$$

The sequence $\{s_n\}$ may or may not be convergent.

It results at once that the coefficients of the expression

$$s_n'' = c_{n1}s_1' + c_{n2}s_2' + \cdots + c_{nn-1}s_{n-1}' + c_{nn}s_n' \quad (12)$$

satisfy a system of linear equations which may be written, after a simple transformation, as follows:

$$\begin{aligned}
c_{n1}\mu_{11} + c_{n2}\mu_{21} + c_{n3}\mu_{31} + \cdots + c_{nn-1}\mu_{n-11} + c_{nn}\mu_{n1} &= \nu_1\mu_{n1} \\
c_{n2}\mu_{22} + c_{n3}\mu_{32} + \cdots + c_{nn-1}\mu_{n-12} + c_{nn}\mu_{n2} &= \nu_2\mu_{n2} \\
c_{n3}\mu_{33} + \cdots + c_{nn-1}\mu_{n-13} + c_{nn}\mu_{n3} &= \nu_3\mu_{n3} \\
\cdots\cdots\cdots

$$
\begin{aligned}
c_{n1} &= \frac{x^2}{1^2 - x^2} \cdot \frac{2^2}{2^2 - x^2} \cdot \frac{3^2}{3^2 - x^2} \cdot \quad \cdots \quad \cdot \frac{n^2}{n^2 - x^2} \\
c_{n2} &= \frac{x^2}{2^2 - x^2} \cdot \frac{3^2}{3^2 - x^2} \cdot \quad \cdots \quad \cdot \frac{n^2}{n^2 - x^2} \\
c_{n3} &= \frac{x^2}{3^2 - x^2} \cdot \quad \cdots \quad \cdot \frac{n^2}{n^2 - x^2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
c_{nn-1} &= \frac{x^2}{(n-1)^2 - x^2} \cdot \frac{n^2}{n^2 - x^2},
\end{aligned}
\tag{15}
$$

for $n = 2, 3, \cdots$, offers no difficulty. Also from (13) we have

$$c_{nn} = \frac{n^2}{n^2 - x^2}. \tag{16}$$

Again, by complete induction we find

$$c_{n1} + c_{n2} + c_{n3} + \cdots + c_{nn} = \frac{1}{x^2} c_{n1}. \tag{17}$$

Therefore

$$\lim_{n \to \infty} (c_{n1} + c_{n2} + c_{n3} + \cdots + c_{nn}) = \frac{\pi x}{\sin \pi x}, \tag{18}$$

and

$$\lim_{n \to \infty} c_{nk} = \frac{x^2}{k^2} \frac{\pi x}{\sin \pi x} \cdot \prod_{j=1}^{k-1} \frac{j^2 - x^2}{j^2}. \tag{19}$$

Finally we may write

$$
\begin{aligned}
c_{nk} &= \frac{x^2}{k^2} \cdot \frac{\pi x}{\sin \pi x} \cdot \prod_{j=1}^{k-1} \left(1 - \frac{x^2}{j^2}\right) \cdot \prod_{j=n+1}^{\infty} \left(1 - \frac{x^2}{j^2}\right) \\
&= \frac{x^2}{k^2} \frac{\pi x}{\sin \pi x} \left\{1 - \sum^{(k, \cdots, n)} \frac{1}{r_1^2} x^2 + \sum^{(k, \cdots, n)} \frac{1}{r_1^2 r_2^2} x^4 - \cdots\right\},
\end{aligned}
\tag{20}
$$

the summation being extended over all positive integers except the integers $(k, k+1, \cdots, n)$.

Now, for every k, and for every n, $n > k$,

$$
\begin{aligned}
&\left|1 - \sum^{(k, \cdots, n)} \frac{1}{r_1^2} x^2 + \sum^{(k, \cdots, n)} \frac{1}{r_1^2 r_2^2} x^4 - \cdots\right| \\
&< \left|1 + \sum_1^{\infty} \frac{1}{r_1^2} |x|^2 + \sum_1^{\infty} \frac{1}{r_1^2 r_2^2} |x|^4 + \cdots\right|.
\end{aligned}
\tag{21}
$$

Hence, for fixed x, not a positive or negative integer, or zero,

$$|c_{nk}| < \frac{1}{k^2} K,$$

$$|c_{n1}| + |c_{n2}| + |c_{n3}| + \cdots + |c_{nn-1}| < \frac{\pi^2}{6} K, \qquad (22)$$

and since by (16), $|c_{nn}|$ is bounded for fixed x,

$$|c_{n1}| + |c_{n2}| + |c_{n3}| + \cdots + |c_{nn-1}| + |c_{nn}| < M, \quad (23)$$

where M depends on x, but not on n.

Thus the constants in (12) satisfy the conditions of the given theorem, and it is thus proved that existence of the first of the limits (7) always implies existence of the second.

In case the limits (5), and consequently the limits (6), exist we write

$$(VP) \sum (-1)^{j-1} \frac{j^2}{j^2 - x^2} \left\{ \frac{2G(j)}{j} x + \frac{2F(j)}{j^2} x^2 \right\} = \xi(x), \quad (24)$$

$$(VP) \sum (-1)^{j-1} \left\{ \frac{2G(j)}{j} x + \frac{2F(j)}{j^2} x^2 \right\} = a_1 x + a_2 x^2. \quad (25)$$

Then, by use of (18) and (19), we have from the given theorem

$$\xi(x) = \frac{\pi x}{\sin \pi x} (a_1 x + a_2 x^2)$$

$$+ \frac{\pi x}{\sin \pi x} \sum_{k=1}^{\infty} \frac{x^2}{k^2} \prod_{j=1}^{k-1} \frac{j^2 - x^2}{j^2} \cdot (a_1^{(k)} - a_1) x \qquad (26)$$

$$+ \frac{\pi x}{\sin \pi x} \sum_{k=1}^{\infty} \frac{x^2}{k^2} \prod_{j=1}^{k-1} \frac{j^2 - x^2}{j^2} \cdot (a_2^{(k)} - a_2) x^2.$$

Then, from (6),

$$\lim_{n \to \infty} P^{(\alpha_n)}(x) = f(0) + \frac{\sin \pi x}{\pi x} \xi(x). \qquad (27)$$

By reference to (26) this limit is readily written in the form, absolutely convergent for every x,

$$\begin{aligned}\Phi(x) = f(0) &+ a_1x + a_2x^2 \\ &- \sum_{j=1}^{\infty} \alpha_{1j}(a_1 - a_1^{(j)})x^3 - \sum_{j=1}^{\infty} \alpha_{1j}(a_2 - a_2^{(j)})x^4 \\ &+ \sum_{j=2}^{\infty} \alpha_{2j}(a_1 - a_1^{(j)})x^5 + \sum_{j=2}^{\infty} \alpha_{2j}(a_2 - a_2^{(j)})x^6 \\ &- \sum_{j=3}^{\infty} \alpha_{3j}(a_1 - a_1^{(j)})x^7 - \sum_{j=3}^{\infty} \alpha_{3j}(a_2 - a_2^{(j)})x^8 \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\end{aligned} \tag{28}$$

where α_{ij} has the value already defined. If, moreover, for fixed n, we approximate $P^{(2n)}(x)$ by the sequence

$$P^{(2)}(x),\ P^{(4)}(x),\ \cdots,\ P^{(2n)}(x),\ P^{(2n)}(x),\ \cdots, \tag{29}$$

we obtain, from (28),

$$\begin{aligned}P^{(2n)}(x) = f(0) &+ a_1^{(n)}x + a_2^{(n)}x^2 \\ &- \sum_{j=1}^{n-1} \alpha_{1j}(a_1^{(n)} - a_1^{(j)})x^3 - \sum_{j=1}^{n-1} \alpha_{1j}(a_2^{(n)} - a_2^{(j)})x^4 \\ &+ \sum_{j=2}^{n-1} \alpha_{2j}(a_1^{(n)} - a_1^{(j)})x^5 + \sum_{j=2}^{n-1} \alpha_{2j}(a_2^{(n)} - a_2^{(j)})x^6 \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &+ (-1)^{n-1}\alpha_{n-1n-1}(a_1^{(n)} - a_1^{(n-1)})x^{2n-1} \\ &+ (-1)^{n-1}\alpha_{n-1n-1}(a_2^{(n)} - a_2^{(n-1)})x^{2n}.\end{aligned} \tag{30}$$

It is worthy of note at this point that, through (4), formula (30) expresses the coefficients of $P^{(2n)}(x)$ as linear functions of the given values $f(-n), \cdots, f(0), \cdots, f(n)$, and is therefore identical with forms of $P^{(2n)}(x)$ previously given by the author,[7] and is not dependent for its validity on existence of the limits by means of which it was derived.

From (28) and (30) it follows that, in case of convergence of $a_1^{(n)}$, and $a_2^{(n)}$, and hence of $P^{(2n)}(x)$,

$$\begin{aligned}\Phi(x) - P^{(2n)}(x) = &(a_1 - a_1^{(n)})x + (a_2 - a_2^{(n)})x^2 \\ &- \sum_{j=1}^{n-1} \alpha_{1j}(a_1 - a_1^{(n)})x^3 - \sum_{j=1}^{n-1} \alpha_{1j}(a_2 - a_2^{(n)})x^4\end{aligned}$$

[7] Loc. cit. (*Trans. A.M.S.*), vol. 26, p. 114, formula (3), and vol. 31, p. 807, formula (2), and p. 820, formula (39).

$$
\begin{aligned}
&+\sum_{j=2}^{n-1}\alpha_{2j}(a_1-a_1^{(n)})x^5+\sum_{j=2}^{n-1}\alpha_{2j}(a_2-a_2^{(n)})x^6\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\\
&+(-1)^{n-1}\alpha_{n-1\,n-1}(a_1-a_1^{(n)})x^{2n-1}\\
&\qquad\qquad+(-1)^{n-1}\alpha_{n-1\,n-1}(a_2-a_2^{(n)})x^{2n}\\
&-\sum_{j=n}^{\infty}\alpha_{1j}(a_1-a_1^{(j)})x^3-\sum_{j=n}^{\infty}\alpha_{1j}(a_2-a_2^{(j)})x^4\\
&+\sum_{j=n}^{\infty}\alpha_{2j}(a_1-a_1^{(j)})x^5+\sum_{j=n}^{\infty}\alpha_{2j}(a_2-a_2^{(j)})x^6\\
&-\sum_{j=n}^{\infty}\alpha_{3j}(a_1-a_1^{(j)})x^7-\sum_{j=n}^{\infty}\alpha_{3j}(a_2-a_2^{(j)})x^8\\
&+\quad\cdots\cdots\cdots\cdots\cdots,
\end{aligned}
\tag{31}
$$

where we place $\alpha_{mj}=0$, for $j<m$.

Now consider a region of the x-plane bounded by a circle of radius ρ with center at the origin. For x in this region, the sum of absolute values of coefficients of $(a_1-a_1^{(n)}),\cdots,(a_2-a_2^{(n)}),\cdots$, in (31) is less than or equal to

$$
(\rho+\rho^2)\left(1+\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\alpha_{ij}\rho^{2i}\right)=(\rho+\rho^2)\frac{\sinh\pi\rho}{\pi\rho}=M. \tag{32}
$$

Then, given arbitrary $\epsilon>0$, we choose N so that

$$
|a_1-a_1^{(n)}|<\frac{\epsilon}{M},\qquad |a_2-a_2^{(n)}|<\frac{\epsilon}{M} \tag{33}
$$

for all values of $n>N$. Then

$$
|\Phi(x)-P^{(2n)}(x)|<\epsilon,\qquad n>N, \tag{34}
$$

uniformly in the given circular region. This completes the proof of the first part of the main theorem.

The proof of the converse is also by application of the given extension of the theorem of Toeplitz. We wish to show that the existence of the second of the limits (7) for any one non-integral x, which is also not zero, always implies existence of the first of these limits.

We shall have proved this if we prove that existence of the first

of the following limits,

$$(VP)\ \sum v_j, \qquad (VP)\ \sum \frac{1}{j^2} v_j \tag{35}$$

always implies existence of the second. We again obtain the equations (13) with $v_j = 1/j^2$.

It is readily proved by complete induction that

$$\begin{aligned}
&\frac{1}{1^2}\frac{\binom{2}{0}}{\binom{2}{1}} + \frac{1}{2^2}\frac{\binom{4}{1}}{\binom{4}{2}} + \frac{1}{3^2}\frac{\binom{6}{2}}{\binom{6}{3}} \\
&\qquad + \cdots + \frac{1}{n^2}\frac{\binom{2n}{n-1}}{\binom{2n}{n}} = \frac{1}{1^2}\frac{\binom{2n}{n-1}}{\binom{2n}{n}} \\
&\frac{1}{2^2}\frac{\binom{4}{0}}{\binom{4}{2}} + \frac{1}{3^2}\frac{\binom{6}{1}}{\binom{6}{3}} + \cdots + \frac{1}{n^2}\frac{\binom{2n}{n-2}}{\binom{2n}{n}} = \frac{1}{2^2}\frac{\binom{2n}{n-2}}{\binom{2n}{n}} \\
&\qquad \frac{1}{3^2}\frac{\binom{6}{0}}{\binom{6}{3}} + \cdots + \frac{1}{n^2}\frac{\binom{2n}{n-3}}{\binom{2n}{n}} = \frac{1}{3^2}\frac{\binom{2n}{n-3}}{\binom{2n}{n}} \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned} \tag{36}$$

Consequently, c_{nk} is equal to $1/k^2$, independently of n. Thus the constants c_{nk} satisfy the conditions of the given theorem in this case also. From (36) we have directly

$$\sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} \frac{1}{j^2} v_j = \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{\binom{2k}{k-j}}{\binom{2k}{k}} v_j. \tag{37}$$

It is evident that, as the theorem requires, the right hand side of (37) has a limit if $\sum v_j$ is (VP) summable, for then we have

$$\sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} \frac{1}{j^2} v_j = \sum_{k=1}^{n} \frac{1}{k^2} \cdot M_k, \tag{38}$$

where M_k is bounded. Finally,

$$(VP) \sum \frac{1}{j^2} v_j = \sum_{k=1}^{\infty} \frac{1}{k^2} M_k, \tag{39}$$

We have now proved that if $P^{(2n)}(x)$ converges for any one value of x which is not a positive or negative integer or zero, the limits $a_1^{(n)} \to a_1$ and $a_2^{(n)} \to a_2$ exist and $P^{(2n)}(x)$ converges to $\Phi(x)$ uniformly in every finite region. The theorem of section 1 is now completely proved.

3. Applications. We shall give two simple examples of integral functions which may be expressed in power series form by means of (28). First, let $P^{(2n)}(x)$ be determined by $f(-j) = -1$, $f(j) = +1, f(0) = 0$.

The limit of $P^{(2n)}(x)$ is then

$$\begin{aligned} G(x) &= \frac{\sin \pi x}{\pi x} (VP) \cdot \sum (-1)^{j-1} \frac{2j}{j^2 - x^2} x \\ &= \frac{\sin \pi x}{\pi x} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{2j}{j^2 - x^2} x \\ &= \frac{\sin \pi x}{\pi x} \sum_{j=1}^{\infty} (-1)^{j-1} \left(\frac{1}{j+x} + \frac{1}{j-x} \right) x. \end{aligned} \tag{40}$$

From a well known expansion of $\psi(x) = \log \Gamma(x)$ we have

$$\begin{aligned} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j+x} &= -\psi(x) + \psi\left(\frac{x}{2}\right) + \frac{1}{x} + \log 2, \\ \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j-x} &= -\psi(-x) + \psi\left(-\frac{x}{2}\right) - \frac{1}{x} + \log 2. \end{aligned} \tag{41}$$

Then $G(x)$ may be expressed in the form

$$G(x) = \frac{\sin \pi x}{\pi x} \left\{ -\psi(x) - \psi(-x) + \psi\left(\frac{x}{2}\right) + \psi\left(-\frac{x}{2}\right) + 2 \log 2 \right\} x. \tag{42}$$

From (4) we have for the sequence of coefficients of x in $P^{(2n)}(x)$, $n = 1, 2, 3, \cdots$,

$$a_1^{(n)} = 2\sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} (-1)^{j-1}\frac{1}{j}. \tag{43}$$

If in formula (37) we substitute $v_j = (-1)^{j-1}j$, we obtain $a_1^{(n)}$ in the form

$$\frac{1}{2}a_1^{(n)} = \sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} \frac{1}{j^2}(-1)^{j-1}j$$

$$= \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{\binom{2k}{k-j}}{\binom{2k}{k}} (-1)^{j-1}j. \tag{44}$$

But we have [8]

$$\frac{\binom{2k}{k-1}\binom{1}{1}}{\binom{2k}{k}} - \frac{\binom{2k}{k-2}\binom{2}{1}}{\binom{2k}{k}}$$

$$+ \cdots + (-1)^{k-1}\frac{\binom{2k}{0}\binom{k}{1}}{\binom{2k}{k}} = \frac{1}{2}\,\frac{\cdot\ k}{2k-1}, \tag{45}$$

and hence

$$a_1^{(n)} = \sum_{k=1}^{n} \frac{1}{k(2k-1)}. \tag{46}$$

Thus the sequence in terms of which $G(x)$ is to be expanded by formula (28) is

$$\begin{aligned} a_1^{(1)} &= \frac{1}{1\cdot 1}, \\ a_1^{(2)} &= \frac{1}{1\cdot 1} + \frac{1}{2\cdot 3}, \\ a_1^{(3)} &= \frac{1}{1\cdot 1} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5}, \\ &\cdots\cdots\cdots, \end{aligned} \tag{47}$$

[8] Netto, Combinatorik, p. 255, formula (42).

from which we have

$$G(x) = (2\log 2)x - \sum_{j=1}^{\infty} \alpha_{1j}(2\log 2 - a_1^{(j)})x^3 + \sum_{j=2}^{\infty} \alpha_{2j}(2\log 2 - a_1^{(j)})x^5 - \sum_{j=3}^{\infty} \alpha_{3j}(2\log 2 - a_1^{(j)})x^7 \quad (48)$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .,$$

or, by substitution from (47),

$$G(x) = (2\log 2)x - \left\{ \frac{1}{2\cdot 3}\cdot\frac{1}{1^2} + \frac{1}{3\cdot 5}\sum_1^2 \frac{1}{r_1^2} + \frac{1}{4\cdot 7}\sum_1^3 \frac{1}{r_1^2} + \cdots \right\} x^3 + \left\{ \frac{1}{3\cdot 5}\cdot\frac{1}{1^2 2^2} + \frac{1}{4\cdot 7}\sum_1^3 \frac{1}{r_1^2 r_2^2} + \cdots \right\} x^5 \quad (49)$$
$$- \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

The expansion of $G(x)$ in Stirling's series,

$$G(x) = \frac{1}{2\cdot 1}\frac{\binom{2}{1}}{1!}x - \frac{1}{2\cdot 3}\frac{\binom{4}{2}}{3!}x(x^2-1) + \frac{1}{2\cdot 5}\frac{\binom{6}{3}}{5!}x(x^2-1)(x^2-2^2) - \frac{1}{2\cdot 7}\frac{\binom{8}{4}}{7!}x(x^2-1)(x^2-2^2)(x^2-3^2) + \cdots, \quad (50)$$

admits arrangement in power series form in agreement with (49).

For a second example, let $P^{(2n)}(x)$ be determined by $f(-j) = +1, f(j) = +1, f(0) = 0$.

The limit of $P^{(2n)}(x)$ is then

$$F(x) = \frac{\sin \pi x}{\pi x}(VP)\sum (-1)^{j-1}\frac{2}{j^2 - x^2}x^2 = \frac{\sin \pi x}{\pi x}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{2}{j^2-x^2}x^2 = \frac{\sin \pi x}{\pi x}\sum_{j=1}^{\infty}(-1)^{j-1}\left(-\frac{1}{j+x}+\frac{1}{j-x}\right)x. \quad (51)$$

Then, by use of (41), $F(x)$ may be expressed in the form

$$F(x) = \frac{\sin \pi x}{\pi x}\left\{\psi(x) - \psi(-x) - \psi\left(\frac{x}{2}\right) + \psi\left(-\frac{x}{2}\right) - \frac{2}{x}\right\} x. \quad (52)$$

By use of the identity

$$\psi(-x) - \psi(x) = \pi \cot \pi x + \frac{1}{x} \quad (53)$$

the expression (52) for $F(x)$ becomes

$$F(x) = 1 - \frac{\sin \pi x}{\pi x}, \quad (54)$$

a result which might also have been obtained from

$$\lim_{n \to \infty}\left\{1 - \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) \cdots \left(1 - \frac{x^2}{n^2}\right)\right\} = 1 - \frac{\sin \pi x}{\pi x}. \quad (55)$$

From (4) we have for the sequence of coefficients of x^2 in $P^{(2n)}(x)$, $n = 1, 2, 3, \cdots$,

$$a_2^{(n)} = 2\sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} (-1)^{j-1} \frac{1}{j^2}. \quad (56)$$

If in formula (37) we substitute $v_j = (-1)^{j-1}$, we obtain $a_2^{(n)}$ in the form

$$\frac{1}{2} a_2^{(n)} = \sum_{j=1}^{n} \frac{\binom{2n}{n-j}}{\binom{2n}{n}} \frac{1}{j^2} (-1)^{j-1}$$

$$= \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{\binom{2k}{k-j}}{\binom{2k}{k}} (-1)^{j-1}. \quad (57)$$

But it is well known that, for all values of k,

$$\frac{\binom{2k}{k-1}}{\binom{2k}{k}} - \frac{\binom{2k}{k-2}}{\binom{2k}{k}} + \cdots + (-1)^{k-1} \frac{\binom{2k}{0}}{\binom{2k}{k}} = \frac{1}{2}, \quad (58)$$

and hence

$$a_2^{(n)} = \sum_{k=1}^{n} \frac{1}{k^2}. \tag{59}$$

Thus the sequence in terms of which $F(x)$ is to be expanded by formula (28) is

$$\begin{aligned} a_2^{(1)} &= \frac{1}{1^2}, \\ a_2^{(2)} &= \frac{1}{1^2} + \frac{1}{2^2}, \\ a_2^{(3)} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}, \\ &\cdots\cdots, \end{aligned} \tag{60}$$

from which we have

$$\begin{aligned} F(x) = \frac{\pi^2}{3!}x^2 &- \sum_{j=1}^{\infty} \alpha_{1j}\left(\frac{\pi^2}{3!} - a_2^{(j)}\right)x^4 \\ &+ \sum_{j=2}^{\infty} \alpha_{2j}\left(\frac{\pi^2}{3!} - a_2^{(j)}\right)x^6 - \sum_{j=3}^{\infty} \alpha_{3j}\left(\frac{\pi^2}{3!} - a_2^{(j)}\right)x^8 \\ &+ \cdots\cdots\cdots\cdots, \end{aligned} \tag{61}$$

or, by substitution from (60),

$$\begin{aligned} F(x) = \frac{\pi^2}{3!}x^2 &- \left\{\frac{1}{2^2}\cdot\frac{1}{1^2} + \frac{1}{3^2}\sum_{1}^{2}\frac{1}{r_1^2} + \frac{1}{4^2}\sum_{1}^{3}\frac{1}{r_1^2} + \cdots\right\}x^4 \\ &+ \left\{\frac{1}{3^2}\cdot\frac{1}{1^2 2^2} + \frac{1}{4^2}\sum_{1}^{3}\frac{1}{r_1^2 r_1^2} + \cdots\right\}x^6 \\ &- \cdots\cdots\cdots\cdots, \end{aligned} \tag{62}$$

The expansion of $F(x)$ in Stirling's series,

$$\begin{aligned} F(x) = \frac{\binom{2}{1}}{2!}x^2 &- \frac{\binom{4}{2}}{4!}x^2(x^2-1) \\ &+ \frac{\binom{6}{3}}{6!}x^2(x^2-1)(x^2-2^2) - \cdots, \end{aligned} \tag{63}$$

admits arrangement in power series form in agreement with (62), which is also equivalent to

$$F(x) = \frac{\pi^2}{3!}x^2 - \frac{\pi^4}{5!}x^4 + \frac{\pi^6}{7!}x^6 - \cdots. \tag{64}$$

4. Convergence of Interpolation Series. When the de la Vallée-Poussin summation indicated in (6) reduces to ordinary convergence, as in (40) and (51), the resulting ordinary series is the cardinal series of interpolation [9]

$$\frac{\sin \pi x}{\pi}\left\{\frac{f(0)}{x} + \sum_{j=1}^{\infty}(-1)^j\left(\frac{f(-j)}{x+j} + \frac{f(j)}{x-j}\right)\right\}. \tag{65}$$

This series does not in general converge when $P^{(2n)}(x)$ is convergent.

On the other hand, the Stirling series of interpolation, at the stage of summation

$$f(0) + \sum_{s=0}^{n-1}(\lambda_s + \lambda_s' x)x(x^2-1)(x^2-2^2)\cdots(x^2-s^2), \tag{66}$$

is *by definition* identical with $P^{(2n)}(x)$. This is equally true of the Gauss series of interpolation, at the stage of summation

$$f(0)+\sum_{s=0}^{n-1}[\mu_s+\mu_s'(x-s-1)]x(x^2-1)(x^2-2^2)\cdots(x^2-s^2). \tag{67}$$

The form (1) of $P^{(2n)}(x)$, and the consequent relation to the cardinal series, has been derived [10] from the Gauss series, but such procedure is unnecessary since the form (1) is an immediate consequence of the classical formula of Lagrange. It has been previously proved [11] that if (66) is convergent for x other than a positive or negative integer or zero, it is uniformly convergent in

[9] See W. L. Ferrar, *Proceedings of the Royal Society of Edinburgh*, vol. 45 (1925), p. 269; also J. M. Whittaker, *Proceedings of the Edinburgh Mathematical Society*, (2), vol. 1 (1927), p. 41.

[10] Ferrar, loc. cit.

[11] Nörlund, *Leçons sur les Series d'Interpolation*, Paris (1926), p. 16.

every finite region to an integral function, a fact contained in the main theorem of this paper.

It is worthy of emphasis that the theorem of section 1 reduces the question of convergence of Stirling's series (or equivalent) to the question of convergence of the sequence of coefficients of x, and of the sequence of coefficients of x^2 (illustrated by (47) and (60) respectively), in any one of the various forms [12] of $P^{(2n)}(x)$ which are arranged according to powers of x. To establish this was the main objective of this paper.

[12] See foot-note 7.

THE APPROXIMATION OF FUNCTIONS AND INTEGRALS BY A LINEAR COMBINATION OF FUNCTIONS *

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1. Introduction. In approximating a function $f(x)$ or an integral $\int_a^b f(x)dx$, the usual procedure is to replace $f(x)$ by any one of several algebraic polynomial approximations. Such polynomial approximations differ only in the data taken to fix them, examples being the interpolation formulas of Lagrange, Newton, Gauss, Stirling, and Bessel;[1] and the integration formulas of Cotes, Gauss, Euler, and Tchebychef.[2] In each of these cases the polynomials are so determined that they or certain of their derivatives coincide with $f(x)$ or certain of its derivatives at a number of given points. General methods and systems of equations have been previously developed by the writer whereby any such approximation formulas may be derived, together with expressions which limit the error; also, the invariant area properties of certain formulas of approximate integration have been investigated. Throughout this development the methods used were explicitly dependent upon the fact that the approximating function is an algebraic polynomial. These methods cannot be used in the case of other approximating functions, for example, in the case of the trigonometric polynomial.

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[1] *La Vallée Poussin: Cours d'Analyse Infinitésimale*, vol. 2, p. 346; *Steffensen: Interpolation; Nörlund: Differenzenrechnung; Gibb: Interpolation and Numerical Integration; Thiele: Interpolationsrechnung.*

[2] *Goursat: Cours d'Analyse Mathématique*, vol. 1, p. 266; *Jordan: Cours d'Analyse*, vol. 2, p. 119; *Radau: Etude sur les formules d'approximation servent à calculer la valeur numérique d'une intégrale définie, Journal de Mathématique*, s. 3, vol. 6, p. 283.

It is the purpose of this paper to show, using other methods, that many of the results referred to above are dependent only upon the fact that the approximating function is linear in terms of the fundamental functions, and not upon the fact that these fundamental functions are the powers of x.

In this paper we consider the approximation of $f(x)$, and in general of $Df(x)$, D being any distributive operator, by means of an approximating function $\zeta(x)$, which is a linear combination of the functions $\omega_1(x)$, $\omega_2(x)$, $\cdots$, $\omega_n(x)$, which are given arbitrarily; thus $\zeta(x) = \sum_{i=1}^{n} a_i\omega_i(x)$, the coefficients a_1, a_2, $\cdots$, a_n being determined so that $f(x)$ and $\zeta(x)$ have certain properties in common. The resulting approximation formulas are placed in the form $\sum_{i=1}^{n} K_iF(c_i)$, where the coefficients K_1, K_2, $\cdots$, K_n are independent of $f(x)$ and may be computed and tabulated once and for all,[3] and where the quantities $F(c_i)$ are dependent only upon $f(x)$. Systems of equations are derived by means of which the above mentioned coefficients may be conveniently computed. Various methods are derived whereby a given approximation formula may be taken as the starting point in the derivation of another of greater complexity. The uniqueness of such approximation formulas is established. The determinateness in any particular case is dependent only upon the nonvanishing of a certain determinant. In all cases, general expressions are derived which limit the error of approximation.

The derivation of integration formulas having invariant area properties is considered, this question being taken up in detail in the case of the algebraic, trigonometric, and hyperbolic polynomials; also, a method is derived whereby the spacings of the ordinates involved in these formulas are fixed by the roots of certain algebraic polynomials, whose derivation is considered.

[3] *Huntington: Tables of Lagrangean Coefficients for Interpolating Without Differences, Proceedings of the American Academy of Arts and Sciences*, vol. 63, p. 421; *Rutledge and Crout: Tables and Methods of Extending Tables for Interpolation Without Differences*, this *Journal*, vol. 9, p. 166.

As extreme cases we have integration formulas analogous to and including those of Gauss, and polynomials analogous to and including those of Legendre.

2. Approximating Functions Fixed by Given Points. Let there be given the n functions $\omega_1(x)$, $\omega_2(x)$, $\cdots$, $\omega_n(x)$; and let us form the function

$$\zeta(x) = a_1\omega_1(x) + a_2\omega_2(x) + \cdots + a_n\omega_n(x), \tag{1}$$

where a_1, a_2, $\cdots$, a_n are constants. Using $\zeta(x)$ as an approximating function, we now determine these constants so that $\zeta(x)$ will coincide with a function $f(x)$ at n given points, c_1, c_2, $\cdots$, c_n; thus we have

$$f(c_i) = \zeta(c_i) = \sum_{j=1}^{n} a_j\omega_j(c_i), \qquad i = 1, 2, \cdots, n. \tag{2}$$

The relations (2) are a system of linear equations whose determinant, which we denote by Δ, will be assumed to be different from zero; thus solving, we have

$$a_j = \frac{1}{\Delta}\sum_{i=1}^{n} A_{i,j} f(c_i), \tag{3}$$

where $A_{i,j}$ is the cofactor of the element of the ith row and jth column of Δ. Substituting (3) in (1), it follows that

$$\zeta(x) = \frac{1}{\Delta}\sum_{j=1}^{n} \omega_j(x) \sum_{i=1}^{n} A_{i,j} f(c_i), \tag{4}$$

which may be written

$$\zeta(x) = \sum_{i=1}^{n} f(c_i) \sum_{j=1}^{n} \frac{A_{i,j}}{\Delta}\, \omega_j(x). \tag{5}$$

We thus have

$$\zeta(x) = \sum_{i=1}^{n} k_i(x) f(c_i), \tag{6}$$

where the coefficients $k_1(x)$, $k_2(x)$, $\cdots$, $k_n(x)$ are independent of $f(x)$, as is shown by the relation

$$k_i(x) = \frac{1}{\Delta}\sum_{j=1}^{n} A_{i,j}\omega_j(x). \tag{7}$$

But $\sum_{j=1}^{n} A_{i,j}\omega_j(x)$ is the determinant formed by replacing the elements of the ith row of Δ with the elements $\omega_1(x)$, $\omega_2(x)$, $\cdots$, $\omega_n(x)$, respectively; thus denoting this determinant by $\Delta_i(x)$, we have

$$k_i(x) = \frac{\Delta_i(x)}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} \omega_1(c_1) & \omega_2(c_1) & \cdots & \omega_n(c_1) \\ \cdot & \cdot & \cdot & \cdot \\ \omega_1(c_{i-1}) & \omega_2(c_{i-1}) & \cdots & \omega_n(c_{i-1}) \\ \omega_1(x) & \omega_2(x) & \cdots & \omega_n(x) \\ \omega_1(c_{i+1}) & \omega_2(c_{i+1}) & \cdots & \omega_n(c_{i+1}) \\ \cdot & \cdot & \cdot & \cdot \\ \omega_1(c_n) & \omega_2(c_n) & \cdots & \omega_n(c_n) \end{vmatrix}. \tag{8}$$

From (8) we see that $k_i(x)$ may be obtained by solving the following system of linear equations: [4, 5]

$$\begin{aligned} k_1\omega_1(c_1) + k_2\omega_1(c_2) + \cdots + k_n\omega_1(c_n) &= \omega_1(x) \\ k_1\omega_2(c_1) + k_2\omega_2(c_2) + \cdots + k_n\omega_2(c_n) &= \omega_2(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ k_1\omega_n(c_1) + k_2\omega_n(c_2) + \cdots + k_n\omega_n(c_n) &= \omega_n(x). \end{aligned} \tag{9}$$

We may observe at this point that if we place

$$\omega_1(x) = 1, \quad \omega_2(x) = x, \quad \cdots, \quad \omega_n(x) = x^{n-1}, \tag{10}$$

[4] The system (9) may also be obtained in the following manner. We place

$$\sum_{j=1}^{n} k_j(x)f(c_j) = \zeta(x) = \sum_{i=1}^{n} a_i\omega_i(x), \tag{I}$$

which may be written

$$\sum_{j=1}^{n} k_j(x) \sum_{i=1}^{n} a_i\omega_i(c_j) = \sum_{i=1}^{n} a_i\omega_i(x), \tag{II}$$

due to (2). But (*II*) may be written

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{n} k_j(x)\omega_i(c_j) = \sum_{i=1}^{n} a_i\omega_i(x); \tag{III}$$

thus for the identity (*III*) to exist, it is necessary and sufficient that

$$\sum_{j=1}^{n} k_j(x)\omega_i(c_j) = \omega_i(x), \qquad i = 1, 2, \cdots, n,$$

which is (9).

[5] In the case where $\zeta(x)$ is an algebraic polynomial, the system (9) has been obtained by different methods. *Crout: A Method for Deriving Formulas of Interpolation*, this *Journal*, vol. 8, p. 18.

then $\zeta(x)$ is an algebraic polynomial of degree $n - 1$; and we obtain the formula of Lagrange.[6]

The relations (6) and (8) may be used directly to determine $\zeta(x)$ as an expression linear in the ordinates of $f(x)$ at the points $c_1, c_2, \cdots, c_n$; however, the following procedure is perhaps simpler, since it reduces the order of the determinants involved. Let us suppose that the approximating function

$$\zeta_{\gamma-1}(x) = \sum_{j=1}^{\gamma-1} a_j \omega_j(x) \tag{11}$$

involving values of $f(x)$ at the points $c_1, c_2, \cdots, c_{\gamma-1}$ is given. It is desired to determine

$$\zeta_\gamma(x) = \sum_{j=1}^{\gamma} b_j \omega_j(x), \tag{12}$$

where the constants $b_1, b_2, \cdots, b_\gamma$ are determined so that

$$\zeta_\gamma(c_i) = f(c_i), \qquad i = 1, 2, \cdots, \gamma;$$

also, it is evident from (4) that but one such expression exists. Let us consider the formula

$$\zeta_{\gamma'}(x) = \sum_{j=1}^{\gamma} d_j \omega_j(x), \tag{13}$$

where the constants $d_1, d_2, \cdots, d_\gamma$ are determined by the relations

$$\begin{aligned} &\zeta_{\gamma'}(c_i) = 0, \qquad i = 1, 2, \cdots, \gamma - 1, \\ &\zeta_{\gamma'}(c_\gamma) = f(c_\gamma) - \zeta_{\gamma-1}(c_\gamma); \end{aligned} \tag{14}$$

then it is at once evident that

$$\zeta_\gamma(x) = \zeta_{\gamma-1}(x) + \zeta_{\gamma'}(x) = d_\gamma \omega_\gamma(x) + \sum_{j=1}^{\gamma-1} (a_j + d_j)\omega_j(x), \tag{15}$$

since $\zeta_\gamma(x)$ is uniquely determinate. From (15) we see that if $\zeta_\gamma(x)$ is determinate, $\zeta_{\gamma'}(x)$ is also determinate.

[6] *Pascal: Die Determinanten*, p. 128.

As an example, in the case of Newton's formula

$$\begin{aligned}\zeta_{\gamma-1}(x) &= m_0 + \sum_{j=1}^{\gamma-2} m_j \prod_{i=1}^{j}(x - c_i),\\ \zeta_{\gamma'}(x) &= m_{\gamma-1} \prod_{i=1}^{\gamma-1}(x - c_i),\\ \zeta_{\gamma}(x) &= m_0 + \sum_{j=1}^{\gamma-1} m_j \prod_{i=1}^{j}(x - c_i),\end{aligned} \tag{16}$$

where $m_0, m_1, m_2, \cdots, m_{\gamma-1}$ are constants.

To this point the functions $\zeta_{\gamma-1}(x)$, $\zeta_{\gamma}(x)$, and $\zeta_{\gamma'}(x)$ have been considered in the form (4); we now consider them in the form (6). In particular we see from (14) that $\zeta_{\gamma'}(x)$ will now consist of the single term

$$\zeta_{\gamma'}(x) = R_{\gamma}(x)[f(c_{\gamma}) - \zeta_{\gamma-1}(c_{\gamma})], \tag{17}$$

where

$$R_{\gamma}(x) = \frac{D_{\gamma,\,x}}{D_{\gamma}}, \tag{18}$$

and

$$\begin{aligned} D_{\gamma,\,x} &= \begin{vmatrix} \omega_1(c_1) & \omega_2(c_1) & \cdots & \omega_{\gamma}(c_1) \\ \omega_1(c_2) & \omega_2(c_2) & \cdots & \omega_{\gamma}(c_2) \\ \cdot & \cdot & \cdots & \cdot \\ \omega_1(c_{\gamma-1}) & \omega_2(c_{\gamma-1}) & \cdots & \omega_{\gamma}(c_{\gamma-1}) \\ \omega_1(x) & \omega_2(x) & \cdots & \omega_{\gamma}(x) \end{vmatrix}, \\ D_{\gamma} &= \begin{vmatrix} \omega_1(c_1) & \omega_2(c_1) & \cdots & \omega_{\gamma}(c_1) \\ \omega_1(c_2) & \omega_2(c_2) & \cdots & \omega_{\gamma}(c_2) \\ \cdot & \cdot & \cdots & \cdot \\ \omega_1(c_{\gamma}) & \omega_2(c_{\gamma}) & \cdots & \omega_{\gamma}(c_{\gamma}) \end{vmatrix}. \end{aligned} \tag{19}$$

From (6) we see that

$$\zeta_{\gamma-1}(c_{\gamma}) = \sum_{i=1}^{\gamma-1} k_i(c_{\gamma}) f(c_i),$$

which substituted in (17) gives

$$\zeta_{\gamma'}(x) = R_{\gamma}(x) f(c_{\gamma}) - \sum_{i=1}^{\gamma-1} R_{\gamma}(x) k_i(c_{\gamma}) f(c_i); \tag{20}$$

thus

$$\zeta_\gamma(x) = \zeta_{\gamma-1}(x) + \zeta_{\gamma'}(x) = R_\gamma(x)f(c_\gamma) + \sum_{i=1}^{\gamma-1} [k_i(x) - R_\gamma(x)k_i(c_\gamma)]f(c_i). \quad (21)$$

From (21) we see that $\zeta_\gamma(x)$ may be obtained from $\zeta_{\gamma-1}(x)$ in the form (6) by subtracting the term $R_\gamma(x)k_i(c_\gamma)$ from the coefficient of $f(c_i)$ and by then adding $R_\gamma(x)f(c_\gamma)$ to the expression thus obtained.

If an approximating function is given which is based upon a particular set of points, it is evident that the above process can be used repeatedly to determine another approximating function based upon these points and a set of additional points; in fact if the given approximating function is based upon a single point, it follows that

$$\zeta_n(x) = \sum_{i=1}^{n} \zeta_{i'}(x), \quad (22)$$

where $\zeta_{1'}(x)$ is defined by the relation

$$\zeta_{1'}(x) = \zeta_1(x) = \frac{f(c_1)}{\omega_1(c_1)}\,\omega_1(x). \quad (23)$$

The relation (22) affords a very convenient means of computing $\zeta_n(x)$, for $\zeta_{i'}(x)$ contains but the single term (17) and involves determinants of order i, whereas in the case of (4) and (6) all determinants involved are of order n. It is evident from (22) that (17) may be written

$$\zeta_{\gamma'}(x) = R_\gamma(x)\Big[f(c_\gamma) - \sum_{i=1}^{\gamma-1} \zeta_{i'}(c_\gamma)\Big], \quad (24)$$

and that

$$R_1(x) = \frac{\omega_1(x)}{\omega_1(c_1)}.$$

It is worthy of remark at this point that if $\zeta_n(x)$ is an algebraic polynomial of degree $n - 1$, then (22) is Newton's formula.

We have already shown that the function $\zeta_{\gamma-1}(x)$ can be

transformed into $\zeta_\gamma(x)$ by the addition of $\zeta_{\gamma'}(x)$; also, the expression (17) offers a convenient method of computing $\zeta_{\gamma'}(x)$. Another method of computing $\zeta_{\gamma'}(x)$ will now be derived which is dependent upon $\zeta_{\gamma-1}(x)$ alone, and which does not involve the use of determinants. Let us define $G_\gamma(x)$ by the relation

$$G_\gamma(x) = \sum_{i=1}^{\gamma} g_i \omega_i(x), \tag{25}$$

where the coefficients $g_1, g_2, \cdots, g_\gamma$ are determined by the conditions

$$\begin{aligned} G_\gamma(c_i) &= 0, \qquad i = 1, 2, \cdots, \gamma - 1, \\ g_\gamma &= 1, \end{aligned} \tag{26}$$

the latter being for convenience alone; then if we place

$$m_\gamma = \frac{f(c_\gamma) - \zeta_{\gamma-1}(c_\gamma)}{G_\gamma(c_\gamma)}, \tag{27}$$

it is at once evident that

$$\zeta_\gamma(x) = \zeta_{\gamma-1}(x) + m_\gamma G_\gamma(x), \tag{28}$$

since $\zeta_\gamma(x)$ is uniquely determinate. It now follows directly that

$$\zeta_{\gamma'}(x) = \frac{[f(c_\gamma) - \zeta_{\gamma-1}(c_\gamma)]}{G_\gamma(c_\gamma)} G_\gamma(x); \tag{29}$$

thus $\zeta_{\gamma'}(x)$ is easily obtained once $G_\gamma(x)$ has been determined. We shall now determine $G_\gamma(x)$ from $\zeta_{\gamma-1}(x)$. Let us apply the approximating function $\zeta_{\gamma-1}(x)$ to the function $\omega_\gamma(x)$, and denote the resulting expression by $\zeta_{\gamma-1}\{x\}$; thus from (6) we see that $\zeta_{\gamma-1}\{x\}$ is of the form

$$\zeta_{\gamma-1}\{x\} = \sum_{i=1}^{\gamma-1} \omega_\gamma(c_i) k_i(x). \tag{30}$$

It is at once evident that

$$G_\gamma(x) = \omega_\gamma(x) - \zeta_{\gamma-1}\{x\}; \tag{31}$$

also, from (29) it follows that

$$\zeta_\gamma(x) = \frac{[f(c_\gamma) - \zeta_{\gamma-1}(c_\gamma)][\omega_\gamma(x) - \zeta_{\gamma-1}\{x\}]}{[\omega_\gamma(c_\gamma) - \zeta_{\gamma-1}\{c_\gamma\}]}; \tag{32}$$

thus $\zeta_\gamma(x)$ is expressed directly in terms of c_γ, $\zeta_{\gamma-1}(x)$, $\omega_\gamma(x)$, and $f(c_\gamma)$, all of which are given. In other words an approximating function $\zeta_{\gamma-1}(x)$ can be used directly to determine a new approximating function $\zeta_\gamma(x)$ when c_γ and $\omega_\gamma(x)$ are given, as is necessarily the case.

Equating (17), (29), and (32), we see that

$$R_\gamma(x) = \frac{G_\gamma(x)}{G_\gamma(c_\gamma)} = \frac{[\omega_\gamma(x) - \zeta_{\gamma-1}\{x\}]}{[\omega_\gamma(c_\gamma) - \zeta_{\gamma-1}\{c_\gamma\}]}. \tag{33}$$

This expression can also be derived directly from (18) using determinants.

3. Certain Interpolation Formulas Fixed by Given Points. We shall now consider a few applications of the theory developed in Section 2. The case where $\zeta(x)$ is an algebraic polynomial of degree $n-1$ has already been taken up in that section. Let us now place $\omega_1(x) = \epsilon^{-\mu ix}$, $\omega_2(x) = \epsilon^{-(\mu-1)ix}$, $\cdots$, $\omega_n(x) = \epsilon^{\mu ix}$, where $i = \sqrt{-1}$; thus

$$\zeta(x) = \sum_{r=-\mu}^{\mu} a_r \epsilon^{irx}, \tag{34}$$

the indices being shifted only for symmetry of notation. The relation (8) thus becomes

$$k_j(x) = \epsilon^{-\mu ic_j} \begin{vmatrix} 1 & \epsilon^{ic_1} & \epsilon^{2ic_1} & \cdots & \epsilon^{2\mu ic_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon^{ic_{j-1}} & \epsilon^{2ic_{j-1}} & \cdots & \epsilon^{2\mu ic_{j-1}} \\ 1 & \epsilon^{ix} & \epsilon^{2ix} & \cdots & \epsilon^{2\mu ix} \\ 1 & \epsilon^{ic_{j+1}} & \epsilon^{2ic_{j+1}} & \cdots & \epsilon^{2\mu ic_{j+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon^{ic_{2\mu+1}} & \epsilon^{2ic_{2\mu+1}} & \cdots & \epsilon^{2\mu ic_{2\mu+1}} \end{vmatrix}$$

$$\div \epsilon^{-\mu ic_j} \begin{vmatrix} 1 & \epsilon^{ic_1} & \epsilon^{2ic_1} & \cdots & \epsilon^{2\mu ic_1} \\ 1 & \epsilon^{ic_2} & \epsilon^{2ic_2} & \cdots & \epsilon^{2\mu ic_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon^{ic_{2\mu+1}} & \epsilon^{2ic_{2\mu+1}} & \cdots & \epsilon^{2\mu ic_{2\mu+1}} \end{vmatrix} \tag{35}$$

after the rows of the numerator and denominator are multiplied by quantities of the form $\epsilon^{i\rho\rho}$, where $\rho = x, c_1, c_2, \cdots, c_{2\mu+1}$. Here it is evident that no two of the values $c_1, c_2, \cdots, c_{2\mu+1}$ can be congruent. The determinants involved in (35) are both of the Vandermonde type; hence we have at once

$$k_j(x) = \frac{\epsilon^{-i\mu x} \prod_{\substack{r=1 \\ r\neq j}}^{2\mu+1} (\epsilon^{ix} - \epsilon^{ic_r})}{\epsilon^{-i\mu c_j} \prod_{\substack{r=1 \\ r\neq j}}^{2\mu+1} (\epsilon^{ic_j} - \epsilon^{ic_r})}; \tag{36}$$

and finally

$$\zeta(x) = \sum_{j=1}^{2\mu+1} \frac{\epsilon^{-i\mu x} \prod_{\substack{r=1 \\ r\neq j}}^{2\mu+1} (\epsilon^{ix} - \epsilon^{ic_r})}{\epsilon^{-i\mu c_j} \prod_{\substack{r=1 \\ r\neq j}}^{2\mu+1} (\epsilon^{ic_j} - \epsilon^{ic_r})} f(c_j), \tag{37}$$

due to (6).

Noting that $\epsilon^{i\alpha} = \cos \alpha + i \sin \alpha$, we see that (34) may be written

$$\zeta(x) = a_0 + \sum_{j=1}^{\mu} [A_j \cos jx + B_j \sin jx], \tag{38}$$

where

$$a_j + a_{-j} = A_j$$
$$ia_j - ia_{-j} = B_j;$$

thus it is evident that (34) and (37) are a trigonometric polynomial of degree μ. Since (37) can be determined directly by (6) and (8) if we place

$$\omega_1(x) = 1, \quad \omega_2(x) = \cos x, \quad \cdots, \quad \omega_{2\mu+1}(x) = \cos \mu x,$$
$$\omega_3(x) = \sin x, \quad \cdots, \quad \omega_{2\mu}(x) = \sin \mu x,$$

and since only real quantities and the four fundamental operations enter into this computation, it is evident that $\zeta(x)$ is real; thus (34) and (37) contain no imaginary part, in spite of the appearance of i in these expressions; furthermore, this is true of the

coefficients $k_j(x)$, (36), since $f(c_j)$ is arbitrary. It is therefore necessary to determine only the real part of (37), this being $\zeta(x)$.

As an example let us consider the case where $\mu = 1$, $c_1 = -\frac{\pi}{2}$, $c_2 = 0$, $c_3 = \frac{\pi}{2}$; then (6) becomes

$$\zeta(x) = \frac{1}{2}[1 - \sin x - \cos x]f\left(-\frac{\pi}{2}\right) + [\cos x]f(0) + \frac{1}{2}[1 + \sin x - \cos x]f\left(\frac{\pi}{2}\right), \quad (39)$$

and (38) is

$$\zeta(x) = \frac{1}{2}\left[f\left(-\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right)\right] + \frac{1}{2}\left[-f\left(-\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right)\right]\sin x + \frac{1}{2}\left[-f\left(-\frac{\pi}{2}\right) + 2f(0) - f\left(\frac{\pi}{2}\right)\right]\cos x.$$

From (36) we have

$$k_3(x) = \frac{\epsilon^{-ix}(\epsilon^{ix} - \epsilon^{-i(\pi/2)})(\epsilon^{ix} - \epsilon^{i0})}{\epsilon^{-i(\pi/2)}(\epsilon^{i(\pi/2)} - \epsilon^{-i(\pi/2)})(\epsilon^{i(\pi/2)} - \epsilon^{i0})} = \frac{\epsilon^{ix} + (i-1) - i\epsilon^{-ix}}{i + (i-1) - 1}$$

$$= \frac{(1 + \sin x - \cos x)(i-1)}{2(i-1)} = \frac{1}{2}[1 + \sin x - \cos x].$$

which agrees with (39). $k_1(x)$ and $k_2(x)$ may be determined in the same way.

We shall now consider the approximating function

$$\zeta(x) = \sum_{r=-\mu}^{\mu} a_r \epsilon^{rx}, \quad (40)$$

which has properties analogous to those of the trigonometric polynomial. Using methods closely analogous to those above, it follows directly that

$$\zeta(x) = a_0 + \sum_{j=1}^{\mu} [A_j \cosh jx + B_j \sinh jx], \quad (41)$$

where

$$a_j + a_{-j} = A_j$$
$$a_j - a_{-j} = B_j;$$

also that

$$k_j(x) = \frac{\epsilon^{-\mu z} \prod_{\substack{r=1 \\ r \neq j}}^{2\mu+1} (\epsilon^z - \epsilon^{c_r})}{\epsilon^{-\mu c_j} \prod_{\substack{r=1 \\ r \neq j}}^{2\mu+1} (\epsilon^{c_j} - \epsilon^{c_r})}, \tag{42}$$

and

$$\zeta(x) = \sum_{j=1}^{2\mu+1} \frac{\epsilon^{-\mu z} \prod_{\substack{r=1 \\ r \neq j}}^{2\mu+1} (\epsilon^z - \epsilon^{c_r})}{\epsilon^{-\mu c_j} \prod_{\substack{r=1 \\ r \neq j}}^{2\mu+1} (\epsilon^{c_j} - \epsilon^{c_r})} f(c_j). \tag{43}$$

4. Symmetrical Systems Fixed by Given Points. We shall now consider the function

$$\zeta(x) = a_{-\mu}\omega_{-\mu}(x) + \cdots + a_{-1}\omega_{-1}(x) + a_1\omega_1(x) + \cdots + a_\mu\omega_\mu(x), \tag{44}$$

symmetrical notation being used for convenience; and let us suppose that

$$\begin{aligned} &\omega_i(c_j) = \omega_i(c_{-j}), && i = 1, 2, \cdots, \mu, \\ &&& j = 1, 2, \cdots, \mu, \\ &\omega_i(c_j) = -\omega_i(c_{-j}), && i = -\mu, -(\mu-1), \cdots, -1, \end{aligned} \tag{45}$$

in which case we say that $\zeta(x)$ is *symmetrical*. We now add the equation of (2) which involves c_{-j} to that which involves c_j, and divide the resulting equation by 2; then we subtract this equation from that which involves c_{-j}. The system (2) is thus broken up into two smaller systems, each consisting of μ unknowns and μ equations. We have

$$\begin{aligned} &\sum_{i=-\mu}^{-1} a_i\omega_i(c_j) = \tfrac{1}{2}[f(c_j) - f(c_{-j})], && j = -\mu, -(\mu-1), \cdots, -1, \\ &\sum_{i=1}^{\mu} a_i\omega_i(c_j) = \tfrac{1}{2}[f(c_j) + f(c_{-j})], && j = 1, 2, \cdots, \mu, \end{aligned} \tag{46}$$

the first set of equations being used to determine $a_{-\mu}, \cdots, a_{-1}$, the second set being used to determine $a_1, \cdots, a_\mu$; thus the order of the determinants involved is reduced from 2μ to μ.

Let us now consider $\zeta(x)$ in the form (6); then the coefficients $k_i(x)$ may be obtained by solving the system (9). Using symmetrical notation we make the substitution

$$\begin{aligned} k_i(x) &= u_i(x) + v_i(x), \\ k_{-i}(x) &= u_i(x) - v_i(x), \end{aligned} \qquad i = 1, 2, \cdots, \mu; \tag{47}$$

then (9) is broken up into two smaller systems; thus in the case of (44), (9) becomes

$$\begin{aligned} &\sum_{j=1}^{\mu} v_j(x)\omega_i(c_j) = \tfrac{1}{2}\omega_i(x), \qquad i = -\mu, -(\mu-1), \cdots, -1, \\ &\sum_{j=1}^{\mu} u_j(x)\omega_i(c_j) = \tfrac{1}{2}\omega_i(x), \qquad i = 1, 2, \cdots, \mu. \end{aligned} \tag{48}$$

As an example let us suppose that $\omega_{-\mu}(x), \cdots, \omega_{-1}(x)$ are odd functions, that $\omega_1(x), \cdots, \omega_\mu(x)$ are even functions, and that

$$c_{-j} = -c_j, \qquad j = 1, 2, \cdots, \mu; \tag{49}$$

then (45) is satisfied, and $\zeta(x)$ is symmetrical. It is evident that this applies directly to the case where $\zeta(x)$ is an algebraic, trigonometric, or hyperbolic polynomial.

Although in (44) $\zeta(x)$ is given as the sum of an even number of terms, it is clear that with certain obvious additions to (45) the methods of this section may be applied in the case where $\zeta(x)$ contains an odd number of terms.

5. Formulas Involving Derivatives of $f(x)$. To this point we have considered approximating functions of the form

$$\zeta(x) = \sum_{i=i_1}^{i_2} k_i(x) f(c_i).$$

We shall now consider functions of the more general type

$$\zeta(x) = \sum_{i=i_{0,1}}^{i_{0,2}} k_{i,0}(x) f(c_{i,0}) + \sum_{i=i_{1,1}}^{i_{1,2}} k_{i,1}(x) f^1(c_{i,1}) + \cdots,$$

or

$$\zeta(x) = \sum_{j=0}^{j_1} \sum_{i=i_{j,1}}^{i_{j,2}} k_{i,j}(x) f^j(c_{i,j}), \tag{50}$$

where $k_{i,j}(x)$ and $c_{i,j}$ are independent of $f(x)$.

Placing

$$\zeta(x) = a_1\omega_1(x) + a_2\omega_2(x) + \cdots + a_n\omega_n(x), \tag{51}$$

we determine the coefficients $a_1, a_2, \cdots, a_n$ so that

$$\begin{aligned}
&\zeta(c_i) = f(c_i), && i = 1, 2, \cdots, n_0,\\
&\zeta^1(c_i) = f^1(c_i), && i = n_0 + 1, \cdots, n_0 + n_1,\\
&\zeta^2(c_i) = f^2(c_i), && i = n_0 + n_1 + 1, \cdots, n_0 + n_1 + n_2,\\
&\cdots\cdots\cdots

where in this case $A_{i,j}$ and Δ refer to the determinant

$$\Delta = \begin{vmatrix} \Omega_1(c_1) & \Omega_2(c_1) & \cdots & \Omega_n(c_1) \\ \Omega_1(c_2) & \Omega_2(c_2) & \cdots & \Omega_n(c_2) \\ \cdot & \cdot & \cdot & \cdot \\ \Omega_1(c_n) & \Omega_2(c_n) & \cdots & \Omega_n(c_n) \end{vmatrix}.$$

Reasoning as with (6), we have

$$\zeta(x) = \sum_{i=1}^{n} k_i(x) F(c_i), \tag{57}$$

where

$$k_i(x) = \frac{1}{\Delta} \sum_{j=1}^{n} A_{i,j} \omega_j(x), \tag{58}$$

or

$$k_i(x) = \frac{1}{\Delta} \begin{vmatrix} \Omega_1(c_1) & \Omega_2(c_1) & \cdots & \Omega_n(c_1) \\ \cdot & \cdot & \cdot & \cdot \\ \Omega_1(c_{i-1}) & \Omega_2(c_{i-1}) & \cdots & \Omega_n(c_{i-1}) \\ \omega_1(x) & \omega_2(x) & \cdots & \omega_n(x) \\ \Omega_1(c_{i+1}) & \Omega_2(c_{i+1}) & \cdots & \Omega_n(c_{i+1}) \\ \cdot & \cdot & \cdot & \cdot \\ \Omega_1(c_n) & \Omega_2(c_n) & \cdots & \Omega_n(c_n) \end{vmatrix}. \tag{59}$$

It is evident that (57) is of the form (50); also, from (59) we see that $k_i(x)$ may be obtained by solving the following system of linear equations:[7,8]

$$\begin{aligned} k_1\Omega_1(c_1) + k_2\Omega_1(c_2) + \cdots + k_n\Omega_1(c_n) &= \omega_1(x) \\ k_1\Omega_2(c_1) + k_2\Omega_2(c_2) + \cdots + k_n\Omega_2(c_n) &= \omega_2(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ k_1\Omega_n(c_1) + k_2\Omega_n(c_2) + \cdots + k_n\Omega_n(c_n) &= \omega_n(x), \end{aligned} \tag{60}$$

which is analogous to (9).

It is evident that relations (56) and (57) may be used directly to determine $\zeta(x)$ if such a function exists; however, we shall now derive certain relations which render it possible to determine

[7] The equations (60) may also be obtained by replacing $\omega_i(x)$ by $\Omega_i(x)$, and $f(c_i)$ by $F(c_i)$ in footnote 4, the reasoning being the same as before.

[8] In the case where $\zeta(x)$ is an algebraic polynomial, the system (60) has been obtained by different methods. See footnote 5.

$\zeta(x)$ in steps, as in the case of (22), or to determine $\zeta(x)$ from some given interpolation function which is more elementary. As in Section 2 let us suppose that $\zeta_{\gamma-1}(x)$ is given in the form (51), where $n = \gamma - 1$, and where the coefficients $a_1, a_2, \cdots, a_{\gamma-1}$ are determined by a set of conditions of the form (53); and let it be desired to determine $\zeta_\gamma(x)$, where $\zeta_\gamma(x)$ is of the form (12), and differs from $\zeta_{\gamma-1}(x)$ in that it involves one more term, $a_\gamma\omega_\gamma(x)$, and satisfies the condition

$$\zeta_\gamma{}^q(c_\gamma) = f^q(c_\gamma) \tag{61}$$

in addition to (53). To this end we define $\zeta_{\gamma'}(x)$ by (12), where the coefficients $d_1, d_2, \cdots, d_\gamma$ are determined so that

$$\begin{aligned}
&\zeta_{\gamma'}(c_i) = 0, \qquad i = 1, 2, \cdots, n_0,\\
&\zeta^1_{\gamma'}(c_i) = 0, \qquad i = n_0 + 1, \cdots, n_0 + n_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\zeta_{\gamma'}{}^m(c_i) = 0, \qquad i = \gamma - n_m, \cdots, \gamma - 1,\\
&\zeta_{\gamma'}{}^q(c_\gamma) = f^q(c_\gamma) - \zeta_{\gamma-1}{}^q(c_\gamma);
\end{aligned}$$

then

$$\zeta_\gamma(x) = \zeta_{\gamma-1}(x) + \zeta_{\gamma'}(x), \tag{62}$$

the reasoning being the same as with (15). In particular if $\zeta_{\gamma'}(x)$ is derived in the form (57), it is necessary to compute but one term; thus

$$\zeta_{\gamma'}(x) = R_\gamma(x)[f^q(c_\gamma) - \zeta_{\gamma-1}{}^q(c_\gamma)], \tag{63}$$

where

$$R_\gamma(x) = \frac{D_{\gamma,x}}{D_\gamma}, \tag{64}$$

and

$$D_{\gamma,x} = \begin{vmatrix}
\Omega_1(c_1) & \Omega_2(c_1) & \cdots & \Omega_\gamma(c_1)\\
\Omega_1(c_2) & \Omega_2(c_2) & \cdots & \Omega_\gamma(c_2)\\
\cdots & \cdots & \cdots & \cdots\\
\Omega_1(c_{\gamma-1}) & \Omega_2(c_{\gamma-1}) & \cdots & \Omega_\gamma(c_{\gamma-1})\\
\omega_1(x) & \omega_2(x) & \cdots & \omega_\gamma(x)
\end{vmatrix},$$

$$D_\gamma = \begin{vmatrix}
\Omega_1(c_1) & \Omega_2(c_1) & \cdots & \Omega_\gamma(c_1)\\
\Omega_1(c_2) & \Omega_2(c_2) & \cdots & \Omega_\gamma(c_2)\\
\cdots & \cdots & \cdots & \cdots\\
\Omega_1(c_\gamma) & \Omega_2(c_\gamma) & \cdots & \Omega_\gamma(c_\gamma)
\end{vmatrix}. \tag{65}$$

Here it is evident that

$$\Omega_j(c_\gamma) = \omega_j{}^q(c_\gamma), \qquad j = 1, 2, \cdots, \gamma,$$

in addition to (53), where $n = \gamma - 1$ and where $j = 1, 2, \cdots, \gamma$; also, we see that (63) is analogous to (17). Combining (57) and (63) we have

$$\zeta_{\gamma'}(x) = R_\gamma(x) f^q(c_\gamma) - \sum_{i=1}^{\gamma-1} R_\gamma(x) k_i{}^q(c_\gamma) F(c_i);$$

thus

$$\zeta_\gamma(x) = R_\gamma(x) f^q(c_\gamma) + \sum_{i=1}^{\gamma-1} [k_i(x) - R_\gamma(x) k_i{}^q(c_\gamma)] F(c_i). \tag{66}$$

From (66) we see that $\zeta_\gamma(x)$ may be obtained from the expression $\zeta_{\gamma-1}(x)$ in the form (57) by subtracting the term $R_\gamma(x) k_i{}^q(c_\gamma)$ from the coefficient of $F(c_i)$ and by then adding $R_\gamma(x) f^q(c_\gamma)$ to the expression thus obtained.

It is evident that (62) can be used repeatedly to give an expression of the form (22), where

$$\zeta_{1'}(x) = \zeta_1(x) = \frac{F(c_1)}{\Omega_1(c_1)} \omega_1(x); \tag{67}$$

thus $\zeta_{\gamma'}(x)$ may be written

$$\zeta_{\gamma'}(x) = R_\gamma(x) [f^q(c_\gamma) - \sum_{i=1}^{\gamma-1} \zeta_i{}^q(c_\gamma)], \tag{68}$$

where

$$R_1(x) = \frac{\omega_1(x)}{\Omega_1(c_1)}.$$

As an example let us consider Maclaurin's series; then $\omega_\gamma(x) = x^{\gamma-1}$, $c_\gamma = 0$, $q = \gamma - 1$, and from (65) we have

$$R_\gamma(x) = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2! & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3! & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \overline{\gamma-2}! & 0 \\ 1 & x & x^2 & x^3 & \cdots & x^{\gamma-2} & x^{\gamma-1} \end{vmatrix}$$

$$\div \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2! & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3! & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \overline{\gamma-2}! & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \overline{\gamma-1}! \end{vmatrix},$$

or

$$R_\gamma(x) = \frac{x^{\gamma-1}}{\overline{\gamma-1}!}, \qquad \sum_{i=1}^{\gamma-1} \zeta_i{}^q(x) = 0;$$

and

$$\zeta_{\gamma'}(x) = \frac{x^{\gamma-1} f^{\gamma-1}(0)}{\overline{\gamma-1}!};$$

thus finally

$$\zeta(x) = f(0) + \sum_{\gamma=1}^{n-1} \frac{x^\gamma f^\gamma(0)}{\gamma!}, \tag{69}$$

as was expected.

We now define $G_\gamma(x)$ by the expression (25), where the coefficients are determined so that

$$\begin{aligned} G_\gamma(c_i) &= 0, \qquad i = 1, 2, \cdots, n_0, \\ G_\gamma{}^1(c_i) &= 0, \qquad i = n_0 + 1, \cdots, n_0 + n_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ G_\gamma{}^m(c_i) &= 0, \qquad i = \gamma - n_m, \cdots, \gamma - 1, \\ g_\gamma &= 1; \end{aligned} \tag{70}$$

then (28) holds true if m_γ is fixed by the relation

$$m_\gamma = \frac{f^q(c_\gamma) - \zeta_{\gamma-1}{}^q(c_\gamma)}{G_\gamma{}^q(c_\gamma)}, \tag{71}$$

the reasoning being similar; thus

$$\zeta_{\gamma'}(x) = \frac{[f^q(c_\gamma) - \zeta_{\gamma-1}{}^q(c_\gamma)]}{G_\gamma{}^q(c_\gamma)} G_\gamma(x). \tag{72}$$

It is evident that (31) is still true; thus (72) may be written

$$\zeta_{\gamma'}(x) = \frac{[f^q(c_\gamma) - \zeta_{\gamma-1}{}^q(c_\gamma)][\omega_\gamma(x) - \zeta_{\gamma-1}\{x\}]}{[\omega_\gamma{}^q(c_\gamma) - \zeta_{\gamma-1}{}^q\{c_\gamma\}]}, \tag{73}$$

which expression gives $\zeta_\gamma(x)$ directly in terms of c_γ, $\zeta_{\gamma-1}(x)$, $\omega_\gamma(x)$, and $f^q(c_\gamma)$, all of which are given. In other words an approximating function $\zeta_{\gamma-1}(x)$ can be used to determine another approximating function $\zeta_\gamma(x)$ when c_γ and $\omega_\gamma(x)$ are given, as is necessarily the case.

Equating (63), (72), and (73), we see that

$$R_\gamma(x) = \frac{G_\gamma(x)}{G_\gamma{}^q(c_\gamma)} = \frac{[\omega_\gamma(x) - \zeta_{\gamma-1}\{x\}]}{[\omega_\gamma{}^q(c_\gamma) - \zeta_{\gamma-1}{}^q\{c_\gamma\}]}, \tag{74}$$

which relation can also be established directly from (64) and (65) by means of determinants.

6. General Case of Symmetrical Systems. In Section 4 it was shown that under certain conditions the determination of $\zeta(x)$ can be simplified considerably. We now extend these methods. Let us suppose that n is even, and that

$$\begin{aligned} \Omega_i(c_{j_1}) &= -\Omega_i(c_{j_2}), \qquad i = 1, 2, \cdots, \frac{n}{2}, \\ \Omega_i(c_{j_1}) &= \Omega_i(c_{j_2}), \qquad i = \left(\frac{n}{2}+1\right), \left(\frac{n}{2}+2\right), \cdots, n; \end{aligned} \tag{75}$$

then the equations of (55) which involve c_{j_1} and c_{j_2}, respectively, may be operated upon in the same manner as each pair of equations in the case of (45), the result being that these two equations may be replaced by the equations

$$\begin{aligned} \sum_{i=1}^{n/2} a_i\Omega_i(c_{j_1}) &= \tfrac{1}{2}[F(c_{j_1}) - F(c_{j_2})] \\ \sum_{i=(n/2)+1}^{n} a_i\Omega_i(c_{j_2}) &= \tfrac{1}{2}[F(c_{j_1}) + F(c_{j_2})], \end{aligned} \tag{76}$$

each of which involves but $\frac{n}{2}$ unknowns. It is thus seen that the work of determining $\zeta(x)$ is simplified.

In the case where n is odd this same method may be used if certain obvious additions are made to (75).

An important application of this principle arises in the case where $n_0, n_1, \cdots, n_m$ are all even, and where each set of equations

of (55) can be broken up into pairs of equations each of which satisfies (75); for example, in the case where (75) is satisfied, and, in order,

$$\begin{aligned} j_1 &= \alpha + 1, \quad \alpha + 2, \quad \cdots, \quad \alpha + \frac{n_k}{2}, \\ j_2 &= \beta - 1, \quad \beta - 2, \quad \cdots, \quad \beta - \frac{n_k}{2}. \end{aligned} \tag{77}$$

where

$$\alpha = \sum_{i=0}^{k-1} n_i, \qquad \beta = \alpha + n_k + 1,$$

and $k = 0, 1, 2, \cdots, m$, in order. Then it is evident that the system (55) can be broken up into two separate systems, each of which involves but $\frac{n}{2}$ unknowns and $\frac{n}{2}$ equations. Similar methods may be used in the case where one or more of the quantities $n_0, n_1, \cdots, n_m$ are odd.

Let us now consider $\zeta(x)$ in the form (57); then the coefficients $k_i(x)$ may be obtained by solving the system (60). If there exists a relation of the form (75), then the substitution

$$\begin{aligned} k_{j_1}(x) &= u_{j_1}(x) + v_{j_1}(x) \\ k_{j_2}(x) &= u_{j_1}(x) - v_{j_1}(x) \end{aligned} \tag{78}$$

will reduce the number of unknowns of the equations involving $k_{j_1}(x)$ and $k_{j_2}(x)$, by one. If there are several relations of the form (75), then several substitutions of the form (78) can be made, the system (60) being thus simplified. In the extreme case, (77), we see that (60) may be broken up into two separate and smaller systems, one involving certain u's and v's, the other the remaining u's and v's. If one or more of the quantities $n_0, n_1, \cdots, n_m$ are odd, (60) can still be broken up into two systems if there results two sets of equations involving the same unknowns.

Relations of the form (75) arise in particular when $\Omega_i(x)$ is odd, or even, and when c_{j_1} and c_{j_2} are symmetrical about the origin. As illustrations we have the algebraic, trigonometric, and hyperbolic polynomials.

7. Case Where Certain Points Approach Coincidence. It was noted in Section 5 that the conditions (52) involved $m + 1$ sets of points, it being only necessary that the points of each set be distinct. We shall now investigate the behavior of $\zeta(x)$ when two or more points of the same set approach coincidence; hence to fix ideas let us suppose that $c_j \to c_i$, where c_i and c_j are of the same set, and where $c_j > c_i$. From (52) we see that

$$\begin{aligned} \zeta^k(c_j) &= f^k(c_j) \\ \zeta^k(c_i) &= f^k(c_i), \end{aligned} \tag{79}$$

where k is known; hence from Rolle's theorem we have

$$\zeta^{k+1}(\xi) = f^{k+1}(\xi),$$

where $c_i < \xi < c_j$. In the limit *if a limit exists* we thus have

$$\zeta^{k+1}(c_i) = f^{k+1}(c_i); \tag{80}$$

hence if $\zeta(x)$ is determined by certain conditions and (79), the limit of $\zeta(x)$ as $c_j \to c_i$ is a formula which satisfies these conditions, the second condition of (79), and (80). It has already been shown that such a formula, which we denote by $z(x)$, can be determined directly, no limiting processes being used; hence we should expect the limit of $\zeta(x)$ as $c_j \to c_i$ to exist and be coincident with $z(x)$. That such is actually the case can be shown directly by means of determinants; thus denoting the coefficients of $z(x)$ by $A_1, A_2, \cdots, A_n, K_1, K_2, \cdots, K_n$, and those of $\zeta(x)$ by $a_1, a_2, \cdots, a_n, k_1, k_2, \cdots, k_n$, we see from (55) and (60) that these coefficients may be expressed as quotients of determinants. As $c_j \to c_i$ it is evident that $a_r \to 0/0$, $r = 1, 2, \cdots, n$; thus, applying L'Hospital's rule, we replace elements involving c_j by the derivatives of these elements at c_i, and have identically

$$\lim_{c_j \to c_i} a_r = A_r, \qquad r = 1, 2, \cdots, n;$$

hence

$$\lim_{c_j \to c_i} k_r = K_r, \qquad r = 1, 2, \cdots, n, r \neq i, j,$$

since (56) and (57) are identically equal. We thus see that

$$\lim_{\epsilon_j \to \epsilon_j} \zeta(x) = z(x).$$

There is evidently no difficulty to be encountered in deriving analogous relations when groups of points of one or more sets approach common limits. For example, if $\zeta(x)$ is an algebraic polynomial, and if all of the points of $\zeta(x)$ approach a common limit; then $\zeta(x)$ becomes a Taylor's expansion of $f(x)$ about that point.

8. Formulas Involving a Distributive Operator. The theory developed in Section 5 involved the process of differentiation; however, the only property of this process which was used was its distributive property. Let us denote by D any distributive operator;[9] then

$$D^\mu \sum_{i=1}^{n} a_i \psi_i(x) = \sum_{i=1}^{n} a_i D^\mu \psi_i(x), \tag{81}$$

where $\psi_1(x), \psi_2(x), \cdots, \psi_n(x)$ are any sequence of functions, and where $a_1, a_2, \cdots, a_n$ are constants. If, now, we replace the definition (54) by the following definition:

$$\psi^i(x) = D^i\psi(x), \tag{82}$$

then the theory developed in Section 5 applies in every detail to the derivation of formulas involving a distributive operator; similarly, if we replace (54) by the definition

$$\psi^{i,\,j}(x) = D_j{}^i\psi(x), \qquad j = 1, 2, \cdots, s, \tag{83}$$

the theory developed in Section 5 applies in full to the derivation of formulas involving the distributive operators $D_1, D_2, \cdots, D_s$.

9. Error Involved in Approximating f(x). Let us denote the error of approximation by ϵ, thus

$$f(x) = \zeta(x) + \epsilon(x); \tag{84}$$

[9] Throughout this paper, (81) will be taken as the definition of a *distributive* operator.

and let us suppose that $f(x)$ may be expanded in a series of the form

$$f(x) = \sum_{j=1}^{N} \alpha_j \omega_j(x), \tag{85}$$

where N is finite or infinite. Substituting (85) in (57), we have

$$\epsilon(x) = \sum_{j=1}^{n} \alpha_j \omega_j(x) + \sum_{j=n+1}^{N} \alpha_j \omega_j(x) - \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n} k_i(x) \Omega_j(c_i) - \sum_{i=1}^{n} k_i(x) \sum_{j=n+1}^{N} \alpha_j \Omega_j(c_i), \tag{86}$$

due to (84). From (60) it follows that

$$\sum_{i=1}^{n} k_i(x) \Omega_j(c_i) = \omega_j(x), \qquad j = 1, 2, \cdots, n;$$

hence substituting in (86) we have

$$\epsilon(x) = \sum_{j=n+1}^{N} \alpha_j \omega_j(x) - \sum_{i=1}^{n} k_i(x) \sum_{j=n+1}^{N} \alpha_j \Omega_j(c_i). \tag{87}$$

Since (85) is convergent it is evident that

$$0 \leq \left| \sum_{j=n+1}^{N} \alpha_j \omega_j(x) \right| \leq \epsilon_0(x); \tag{88}$$

also, supposing $f(x)$ to be such that (85) may be operated upon term by term by the distributive operators which give rise to the expressions Ω_j, it follows that

$$0 \leq \left| \sum_{j=n+1}^{N} \alpha_j \Omega_j(c_i) \right| \leq \epsilon_1(c_i), \qquad i = 1, 2, \cdots, n. \tag{89}$$

From (87) we now have finally

$$|\epsilon(x)| \leq \epsilon_0(x) + \sum_{i=1}^{n} |k_i(x)| \epsilon_1(c_i). \tag{90}$$

As an example let us suppose that (85) is an expansion in a power series, and that the operators involved are derivatives of order not greater than γ; then denoting the greatest $|f^{\gamma}(x)|$ in the interval bounded by the two outermost points of the set 0, x, c_i,

$\cdots$, c_n by f_m^i, it follows that

$$\epsilon_0(x) = \frac{x^n f_m^n}{n!},$$

$$\epsilon_1(c_i) = \frac{c_i^{n-\gamma} f_m^n}{n-\gamma!}, \qquad i = 1, 2, \cdots, n;$$

hence (90) becomes

$$|\epsilon(x)| \leq \tau f_m^n, \tag{91}$$

where τ may be computed.

10. Uniqueness of Results. Let us suppose that the system of equations (55) is uniquely determinate, or that $\Delta \neq 0$; then it is evident that there can be but one expression of the form (56). Since Δ is also the determinant of (60), we see that (57) is also uniquely determinate. Again, let us suppose that by some method there has been derived a formula

$$z(x) = \sum_{i=1}^{n} \kappa_i(x) F(c_i),$$

of the form (57), which gives exact results when applied to any function of the form (51); then restricting $f(x)$ to functions of this form, we have

$$\sum_{i=1}^{n} [\kappa_i(x) - k_i(x)] F(c_i) = 0. \tag{92}$$

Since for any set of values of $F(c_i)$, $i = 1, 2, \cdots, n$, (57) may be used to give a function of the form (51), we see that (92) holds true for any set of values of $F(c_i)$, $i = 1, 2, \cdots, n$; hence for any x we have

$$\kappa_i(x) - k_i(x) = 0, \qquad i = 1, 2, \cdots, n;$$

and

$$z(x) = \zeta(x).$$

We thus see that the formula derived by the other method is identical with (57).

11. Formulas for the Approximate Calculation of Integrals. From (84) we see that

$$\int_a^b f(x)dx = \int_a^b \zeta(x)dx + \int_a^b \epsilon(x)dx; \tag{93}$$

thus we consider $\int_a^b \zeta(x)dx$ as an approximation to $\int_a^b f(x)dx$, the error being $\int_a^b \epsilon(x)dx$.

We shall first consider the case where $\zeta(x)$ is fixed by values of $f(x)$ at given points; thus from (6) we have

$$\int_a^b \zeta(x)dx = \sum_{i=1}^{n} K_i f(c_i), \tag{94}$$

where

$$K_i = \int_a^b k_i(x)dx, \qquad i = 1, 2, \cdots, n. \tag{95}$$

Here it is to be noted that the coefficients $K_1, K_2, \cdots, K_n$ are independent of $f(x)$, and may be computed and tabulated once and for all when $a, b, \omega_1(x), \omega_2(x), \cdots, \omega_n(x)$ are given. From (7) we see that

$$K_i = \frac{1}{\Delta}\sum_{j=1}^{n} A_{i,j}\int_a^b \omega_j(x)dx;$$

hence placing

$$I_j = \int_a^b \omega_j(x)dx, \qquad j = 1, 2, \cdots, n, \tag{96}$$

we have,[10] as in the case of (7) and (8),

$$K_i = \frac{1}{\Delta}\begin{vmatrix} \omega_1(c_1) & \omega_2(c_1) & \cdots & \omega_n(c_1) \\ \cdots & \cdots & \cdots & \cdots \\ \omega_1(c_{i-1}) & \omega_2(c_{i-1}) & \cdots & \omega_n(c_{i-1}) \\ I_1 & I_2 & \cdots & I_n \\ \omega_1(c_{i+1}) & \omega_2(c_{i+1}) & \cdots & \omega_n(c_{i+1}) \\ \cdots & \cdots & \cdots & \cdots \\ \omega_1(c_n) & \omega_2(c_n) & \cdots & \omega_n(c_n) \end{vmatrix}. \tag{97}$$

[10] Let us suppose that an nth degree determinant has a row (or column) whose elements are, in order, $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$, all the other elements being constants; then if the distributive operator D operate upon this determinant, the result is obtained by replacing the row (or column) of elements $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ by the row (or column) $D\phi_1(x), D\phi_2(x), \cdots, D\phi_n(x)$. This may be shown by developing the determinant by minors with respect to this row (or column), letting D operate, and by then rebuilding the determinant.

From (97) we see that K_i may be obtained by solving the following system of linear equations: [11]

$$\begin{aligned} K_1\omega_1(c_1) + K_2\omega_1(c_2) + \cdots + K_n\omega_1(c_n) &= I_1 \\ K_1\omega_2(c_1) + K_2\omega_2(c_2) + \cdots + K_n\omega_2(c_n) &= I_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ K_1\omega_n(c_1) + K_2\omega_n(c_2) + \cdots + K_n\omega_n(c_n) &= I_n, \end{aligned} \tag{98}$$

which is analogous to (9), and which may be derived by integrating each equation of (9).

By integrating equations (15), (17) $\cdots$ (22), (24), (29), (32), and (33), we obtain corresponding relations for integration formulas. In particular we note that

$$\int_a^b D_{\gamma,x}dx = \begin{vmatrix} \omega_1(c_1) & \omega_2(c_1) & \cdots & \omega_\gamma(c_1) \\ \omega_1(c_2) & \omega_2(c_2) & \cdots & \omega_\gamma(c_2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \omega_1(c_{\gamma-1}) & \omega_2(c_{\gamma-1}) & \cdots & \omega_\gamma(c_{\gamma-1}) \\ I_1 & I_2 & \cdots & I_\gamma \end{vmatrix}, \tag{99}$$

due to (19) and (96).

We now consider the case where $\zeta(x)$ involves derivatives of $f(x)$. From (57) we have directly

$$\int_a^b \zeta(x)dx = \sum_{i=1}^{n} K_i F(c_i), \tag{100}$$

where

$$K_i = \int_a^b k_i(x)dx, \qquad i = 1, 2, \cdots, n. \tag{101}$$

From (58) we see that

$$K_i = \frac{1}{\Delta}\sum_{j=1}^{n} A_{i,j}\int_a^b \omega_j(x)dx;$$

[11] In the case where $\zeta(x)$ is an algebraic polynomial, the system (98) has been obtained by different methods. *Crout: A Method for Deriving Formulas for the Approximate Calculation of Integrals*, this *Journal*, vol. 7, p. 126; also vol. 8, p. 119.

hence, defining I_i by (96) and reasoning as with (97), we have

$$K_i = \frac{1}{\Delta}\begin{vmatrix} \Omega_1(c_1) & \Omega_2(c_1) & \cdots & \Omega_n(c_1) \\ \cdot & \cdot & \cdots & \cdot \\ \Omega_1(c_{i-1}) & \Omega_2(c_{i-1}) & \cdots & \Omega_n(c_{i-1}) \\ I_1 & I_2 & \cdots & I_n \\ \Omega_1(c_{i+1}) & \Omega_2(c_{i+1}) & \cdots & \Omega_n(c_{i+1}) \\ \cdot & \cdot & \cdots & \cdot \\ \Omega_1(c_n) & \Omega_2(c_n) & \cdots & \Omega_n(c_n) \end{vmatrix}. \tag{102}$$

From (102) we see that K_i may be obtained by solving the following system of linear equations: [12]

$$\begin{aligned} K_1\Omega_1(c_1) + K_2\Omega_1(c_2) + \cdots + K_n\Omega_1(c_n) &= I_1 \\ K_1\Omega_2(c_1) + K_2\Omega_2(c_2) + \cdots + K_n\Omega_2(c_n) &= I_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ K_1\Omega_n(c_1) + K_2\Omega_n(c_2) + \cdots + K_n\Omega_n(c_n) &= I_n, \end{aligned} \tag{103}$$

which is analogous to (60), and which may be derived by integrating each equation of (60).

By integrating equations (62) $\cdots$ (68), (72) $\cdots$ (74), we obtain corresponding relations for integration formulas. In particular we note that

$$\int_a^b D_{\gamma,x}dx = \begin{vmatrix} \Omega_1(c_1) & \Omega_2(c_1) & \cdots & \Omega_\gamma(c_1) \\ \Omega_1(c_2) & \Omega_2(c_2) & \cdots & \Omega_\gamma(c_2) \\ \cdot & \cdot & \cdots & \cdot \\ \Omega_1(c_{\gamma-1}) & \Omega_2(c_{\gamma-1}) & \cdots & \Omega_\gamma(c_{\gamma-1}) \\ I_1 & I_2 & \cdots & I_\gamma \end{vmatrix}, \tag{104}$$

due to (65) and (96).

It is evident that if we replace the definition (54) by (82) or (83), the preceding theory applies in full to the derivation of integration formulas which involve one or more distributive operators.

12. Symmetrical Integration Formulas. In Section 4 a method

[12] In the case where $\zeta(x)$ is an algebraic polynomial, the system (103) has been obtained by different methods. See footnote 11.

was developed whereby the solution of the system (9) is greatly simplified if $\zeta(x)$ is symmetrical. We shall say that the integration formula $\int_a^b \zeta(x)dx$ is *symmetrical* if $\zeta(x)$ is symmetrical; thus comparing (9) and (98) we see that if $\int_a^b \zeta(x)dx$ is symmetrical, we can apply this same method to simplify the solution of (98). Here we make the substitution

$$\begin{aligned} K_i &= U_i + V_i, \\ K_{-i} &= U_i - V_i, \end{aligned} \qquad i = 1, 2, \cdots, \mu, \tag{105}$$

in place of (47); also, from (47) and (105) it is evident that

$$\begin{aligned} U_i &= \int_a^b u_i(x)dx, \\ V_i &= \int_a^b v_i(x)dx, \end{aligned} \qquad i = 1, 2, \cdots, \mu.$$

In an exactly similar manner we can use the methods of Section 6 to simplify the solution of (103); thus in place of (78) we have

$$\begin{aligned} K_{j_1} &= U_{j_1} + V_{j_1}, \\ K_{j_2} &= U_{j_1} - V_{j_1}, \end{aligned} \tag{106}$$

which may be obtained by integrating (78).

13. Error Involved in Approximating $\int_a^b f(x)dx$. From (84) we see that the error involved in approximating $\int_a^b f(x)dx$ is

$$\bar{\epsilon} = \int_a^b \epsilon(x)dx; \tag{107}$$

hence, supposing that $f(x)$ may be expanded in the series (85), that this series is uniformly convergent in the interval (a, b), and that the functions $\omega_1(x), \omega_2(x), \cdots$ are continuous in this interval;

we have, integrating (87) noting (96) and (101),

$$\bar{\epsilon} = \sum_{j=n+1}^{N} \alpha_j I_j - \sum_{i=1}^{n} K_i \sum_{j=n+1}^{N} \alpha_j \Omega_j(c_i). \tag{108}$$

Reasoning as in Section 9, it follows that

$$0 \leq \Big| \sum_{j=n+1}^{N} \alpha_j I_j \Big| \leq \bar{\epsilon}_0; \tag{109}$$

hence, noting (89) we have finally from (108)

$$|\bar{\epsilon}| \leq \bar{\epsilon}_0 + \sum_{i=1}^{n} |K_i| \epsilon_i(c_i). \tag{110}$$

14. Uniqueness of Integration Formulas. Let us suppose that the system of equations (103) is uniquely determinate, or that $\Delta \neq 0$; also, let us suppose that by some other method there has been derived an integration formula

$$J = \sum_{i=1}^{n} \kappa_i F(c_i),$$

of the form (100), which gives exact results when applied to any function of the form (51); then restricting $f(x)$ to functions of this form, we have

$$\sum_{i=1}^{n} [\kappa_i - K_i] F(c_i) = 0, \qquad i = 1, 2, \cdots, n.$$

Since $\Delta \neq 0$, (57) may be used to give a function of the form (51) for any arbitrary set of values of $F(c_i)$, $i = 1, 2, \cdots, n$; hence

$$\kappa_i - K_i = 0, \qquad i = 1, 2, \cdots, n;$$

and

$$J = \int_a^b \zeta(x) dx.$$

We thus see that (100) and the formula derived by the other method are identical.

15. Invariant Area. In Section 11 it was shown that if $\zeta(x)$ is fixed by given points $\int_a^b \zeta(x) dx$ may be expressed in the

form (94), where the quantities $K_1, K_2, \cdots, K_n$ are fixed by the equations (98). Let us now suppose that the equations

$$\begin{aligned} K_1\omega_1(c_1) + K_2\omega_1(c_2) + \cdots + K_m\omega_1(c_m) &= I_1 \\ K_1\omega_2(c_1) + K_2\omega_2(c_2) + \cdots + K_m\omega_2(c_m) &= I_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ K_1\omega_n(c_1) + K_2\omega_n(c_2) + \cdots + K_m\omega_n(c_m) &= I_n, \end{aligned} \tag{111}$$

where $m \leq n$, are consistent and determinate; then (98) is satisfied by the values of the coefficients given by (111) and by the relations

$$K_{m+1} = K_{m+2} = \cdots = K_n = 0; \tag{112}$$

hence, since $\Delta \neq 0$ this is the only solution of (98). (94) now becomes

$$\int_a^b \zeta(x)dx = \sum_{i=1}^{m} K_i f(c_i), \tag{113}$$

where the coefficients $K_1, K_2, \cdots, K_m$ are given by (111). Since $\Delta \neq 0$, n points are required to fix $\zeta(x)$; but from (113) we see that the m points $(c_1, f(c_1)), \cdots, (c_m, f(c_m))$ determine $\int_a^b \zeta(x)dx$; thus it is evident that $\int_a^b \zeta(x)dx$ is invariant for the ∞^{n-m} functions $\zeta(x)$ passing through these points.

Similarly, if $\zeta(x)$ involves derivatives or distributive operators, we suppose that the equations

$$\begin{aligned} K_1\Omega_1(c_1) + K_2\Omega_1(c_2) + \cdots + K_m\Omega_1(c_m) &= I_1 \\ K_1\Omega_2(c_1) + K_2\Omega_2(c_2) + \cdots + K_m\Omega_2(c_m) &= I_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ K_1\Omega_n(c_1) + K_2\Omega_n(c_2) + \cdots + K_m\Omega_n(c_m) &= I_n, \end{aligned} \tag{114}$$

where $m \leq n$, are consistent and determinate; then proceeding from (100), (103), and (114) as with (94), (98), and (111), we obtain the relation

$$\int_a^b \zeta(x)dx = \sum_{i=1}^{m} K_i F(c_i); \tag{115}$$

thus $\int_a^b \zeta(x)dx$ is invariant for the ∞^{n-m} functions $\zeta(x)$ satisfying the conditions

$$\sum_{j=1}^{n} a_j \Omega_j(c_i) = F(c_i), \qquad i = 1, 2, \cdots, m. \tag{116}$$

The conditions which render the systems (111) and (114) consistent and determinate may be expressed as the vanishing of certain determinants [13] which involve $I_1, I_2, \cdots, I_n$ and $\omega_i(c_j)$, or $\Omega_i(c_j)$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$; thus since the functions $\omega_1(x), \omega_2(x), \cdots, \omega_n(x)$, or $\Omega_1(x), \Omega_2(x), \cdots, \Omega_n(x)$ are known, $I_1, I_2, \cdots, I_n$ are known; and these conditions may be treated as conditions subject to which $c_1, c_2, \cdots, c_m$ are initially chosen.

16. Certain Cases of Invariant Area. Let us first consider the case where $\zeta(x)$ is an algebraic polynomial of degree $m + \alpha$; thus

$$\omega_1(x) = 1, \omega_2(x) = x, \cdots, \omega_{m+\alpha+1}(x) = x^{m+\alpha};$$

and (111) becomes

$$\begin{aligned} K_1 + K_2 + \cdots + K_m &= I_1 \\ K_1c_1 + K_2c_2 + \cdots + K_mc_m &= I_2 \\ \cdots\cdots\cdots\

equations, this determinant becomes

$$\delta_\gamma = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_m \\ \cdot & \cdot & \cdot & \cdot \\ c_1^{m-1} & c_2^{m-1} & \cdots & c_m^{m-1} \end{vmatrix} \prod_{j=1}^{m} c_j^{\gamma} \neq 0, \quad \gamma = 0, 1, 2, \cdots, \alpha+1, \quad (119)$$

since the determinant involved in (119) is of the Vandermonde type and the c's are distinct; hence the matrix

$$\left\| \begin{matrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_m \\ \cdot & \cdot & \cdot & \cdot \\ c_1^{m+\alpha} & c_2^{m+\alpha} & \cdots & c_m^{m+\alpha} \end{matrix} \right\|$$

is of rank m. The conditions

$$\begin{vmatrix} c_1^{\gamma} & c_2^{\gamma} & \cdots & c_m^{\gamma} & I_{\gamma+1} \\ c_1^{\gamma+1} & c_2^{\gamma+1} & \cdots & c_m^{\gamma+1} & I_{\gamma+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1^{\gamma+m} & c_2^{\gamma+m} & \cdots & c_m^{\gamma+m} & I_{\gamma+m+1} \end{vmatrix} = 0, \quad \gamma = 0, 1, 2, \cdots, \alpha, \quad (120)$$

may thus be taken as the necessary and sufficient conditions for the equations (117) to be consistent,[13] it being evident from (119) that (117) is determinate. Factoring c_1^{γ} from the first column, c_2^{γ} from the second, $\cdots$, and c_m^{γ} from the mth column of (120), noting (118), we have

$$\int_a^b \begin{vmatrix} 1 & 1 & \cdots & 1 & x^{\gamma} \\ c_1 & c_2 & \cdots & c_m & x^{\gamma+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1^{m} & c_2^{m} & \cdots & c_m^{m} & x^{\gamma+m} \end{vmatrix} dx = 0, \quad \gamma = 0, 1, 2, \cdots, \alpha.$$

We now factor x^{γ} from the last column of the integrand; then noting that the remaining determinant is of the Vandermonde type, we have finally

$$\int_a^b x^{\gamma} \prod_{j=1}^{m} (x - c_j) dx = 0, \qquad \gamma = 0, 1, 2, \cdots, \alpha, \quad (121)$$

which conditions are necessary and sufficient for the integration formula (113) to give exact results when applied to any algebraic polynomial of degree $m + \alpha$.

Had one of the quantities $c_1, c_2, \cdots, c_m$ been zero, c_m to fix ideas, we should have considered groups of $m - 1$ consecutive equations of (117) in exactly the same way as before except that in this case $\gamma = 1, 2, \cdots, \alpha + 2$; hence we should have obtained the necessary and sufficient conditions

$$\int_a^b x^\gamma \prod_{j=1}^{m-1} (x - c_j)dx = 0, \qquad \gamma = 1, 2, \cdots, \alpha + 1.$$

Since $c_m = 0$, we have $x = (x - c_m)$; thus (121) holds true in this case also.

Proceeding from (121), it is possible to determine a family of algebraic polynomials of degree m such that the m roots of each may be taken as a set of values $c_1, c_2, \cdots, c_m$ which satisfies the conditions (121). The extreme case where $n = 2m$ and all of the c's are fixed by (121) leads directly to Legendre's polynomials and to Gauss' formulas of integration.[14, 15]

We shall next consider the case where $\zeta(x)$ is a trigonometric polynomial of degree μ; thus, as with (34), we have

$$\omega_1(x) = \epsilon^{-i\mu x}, \quad \omega_2(x) = \epsilon^{-i(\mu-1)x}, \quad \cdots, \quad \omega_n(x) = \epsilon^{i\mu x};$$

and (111) becomes

$$\begin{aligned} K_1\epsilon^{-i\mu c_1} + K_2\epsilon^{-i\mu c_2} + \cdots + K_m\epsilon^{-i\mu c_m} &= I_{-\mu} \\ \cdots\cdots\cdots\

we now consider a group of m consecutive equations of (122), the first being

$$K_1\epsilon^{i\gamma c_1} + K_2\epsilon^{i\gamma c_2} + \cdots + K_m\epsilon^{i\gamma c_m} = I_\gamma,$$
$$\gamma = -\mu, -(\mu-1), \cdots, (\mu-m+1);$$

and after factoring $\epsilon^{i\gamma c_1}$ from the first column, $\epsilon^{i\gamma c_2}$ from the second, $\cdots$, and $\epsilon^{i\gamma c_m}$ from the last column of the determinant of these m equations, this determinant becomes

$$\delta_\gamma = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \epsilon^{ic_1} & \epsilon^{ic_2} & \cdots & \epsilon^{ic_m} \\ \epsilon^{2ic_1} & \epsilon^{2ic_2} & \cdots & \epsilon^{2ic_m} \\ \cdots & \cdots & \cdots & \cdots \\ \epsilon^{(m-1)ic_1} & \epsilon^{(m-1)ic_2} & \cdots & \epsilon^{(m-1)ic_m} \end{vmatrix} \prod_{j=1}^{m} \epsilon^{i\gamma c_j} \neq 0,$$
$$\gamma = -\mu, -(\mu-1), \cdots, (\mu-m+1), \quad (123)$$

since the determinant involved in (123) is of the Vandermonde type and the c's are distinct. We now proceed as from (119), and finally obtain the conditions

$$\int_a^b \epsilon^{i\gamma x} \prod_{j=1}^{m} (\epsilon^{ix} - \epsilon^{ic_j})dx = 0,$$
$$\gamma = -\mu, -(\mu-1), \cdots, (\mu-m), \quad (124)$$

which are necessary and sufficient for the integration formula (113) to give exact results when applied to any trigonometric polynomial of degree μ.

We shall now derive a family of algebraic polynomials of degree m such that the m roots of each may be taken as a set of ϵ^{ic_j}s which satisfies (124). After making the substitutions

$$\epsilon^{ix} = z, \quad \epsilon^{ic_j} = \beta_j, \quad a_1 = \epsilon^{ia}, \quad b_1 = \epsilon^{ib},$$

(124) becomes

$$\int_{a_1}^{b_1} z^\gamma \prod_{j=1}^{m} (z - \beta_j)dz = 0,$$
$$\gamma = -(\mu+1), -\mu, -(\mu-1), \cdots, (\mu-m-1). \quad (125)$$

Denoting the sum of the β's taken one at a time by s_1, the sum

of the β's taken two at a time by s_2, $\cdots$, and the sum of the β's taken m at a time by s_m, it is evident that (125) may be written

$$\int_{a_1}^{b_1} [z^{\gamma+m} - s_1 z^{\gamma+m-1} + \cdots \pm s_m z^{\gamma}] dz = 0,$$
$$\gamma = -(\mu+1), -\mu, -(\mu-1), \cdots, (\mu-m-1),$$

or

$$s_1 \int_{a_1}^{b_1} z^{\gamma+m-1} dz - s_2 \int_{a_1}^{b_1} z^{\gamma+m-2} dz + \cdots \mp s_m \int_{a_1}^{b_1} z^{\gamma} dz = \int_{a_1}^{b_1} z^{\gamma+m} dz,$$
$$\gamma = -(\mu+1), -\mu, -(\mu-1), \cdots, (\mu-m-1). \quad (126)$$

The relations (126) consist of $(2\mu - m + 1)$ linear equations; thus in addition to (126) we impose the conditions

$$s_1 \alpha_{j,1} + s_2 \alpha_{j,2} + \cdots + s_m \alpha_{j,m} = \alpha_{j,(m+1)}, \quad (127)$$
$$j = 1, 2, \cdots, (2m - 2\mu - 1),$$

and hence obtain a system of m linear equations, which we solve for $s_1, s_2, \cdots, s_m$. We now form the algebraic polynomial

$$u^m - s_1 u^{m-1} + s_2 u^{m-2} - \cdots \pm s_m; \quad (128)$$

then it is evident that the roots of (128) are $\beta_1, \beta_2, \cdots, \beta_m$. Since $s_1, s_2, \cdots, s_m$ are expressed in terms of the parameters $\alpha_{j,k}$, the polynomial (128) involves these same parameters; hence, once the quantities $\alpha_{j,k}$ are fixed, either arbitrarily or in accordance with further conditions, we obtain definite numerical values for $\beta_1, \beta_2, \cdots, \beta_m$. The quantities $c_1, c_2, \cdots, c_m$ are thus determined, for $c_j = -i \log \beta_j$, $j = 1, 2, \cdots, m$. In the particular case where (126) contains $m - 1$ equations, or $m = \mu + 1$, it is evident that the resulting integration formulas are closely analogous to those of Gauss for the algebraic polynomial.

As an example let us suppose that $\mu = 1$, $m = 2$, $-a = b = \pi$; then from (126) we have

$$s_1 \int_{e^{-i\pi}}^{e^{i\pi}} z^{-1} dz - s_2 \int_{e^{-i\pi}}^{e^{i\pi}} z^{-2} dz = \int_{e^{-i\pi}}^{e^{i\pi}} dz,$$

which is

$$s_1 = 0.$$

From (127) it thus follows that

$$s_2\alpha_{1,2} = \alpha_{1,3},$$

or

$$s_2 = \frac{\alpha_{1,3}}{\alpha_{1,2}} = -\alpha^2,$$

where α^2 is arbitrary; thus (128) becomes

$$u^2 - \alpha^2,$$

and

$$\beta_1 = \alpha, \qquad \beta_2 = -\alpha.$$

We now place

$$\alpha = r(\cos\theta + i\sin\theta),$$

where r and θ are real; then

$$c_1 = -i[\log r + i\theta], \qquad c_2 = -i[\log r + i(\theta + \pi)].$$

Restricting α to points of the unit circle, which restriction is necessary if c_1 and c_2 are to be real, it follows that

$$c_1 = \theta, \qquad c_2 = \theta + \pi,$$

θ being arbitrary; also, from (122) we have

$$K_{-1} = K_1 = \pi;$$

thus finally

$$\int_{-\pi}^{\pi} \zeta(x)dx = \pi f(\theta) + \pi f(\theta + \pi) \tag{129}$$

for any real θ. It is evident that (129) gives exact results when applied to any trigonometric polynomial of first degree.[16] As a special case let us place $\theta = -\frac{\pi}{2}$; then (129) becomes

$$\int_{-\pi}^{\pi} \zeta(x)dx = \pi f\left(-\frac{\pi}{2}\right) + \pi f\left(\frac{\pi}{2}\right).$$

[16] Since restrictions have been imposed arbitrarily, and since the principal values of the logarithm have been used, it is evident that (129) is not the most general integration formula having this property.

It is evident that if we replace i by 1 in (122) $\cdots$ (128), these formulas together with their derivations are transformed to the case where $\zeta(x)$ is a hyperbolic polynomial of degree μ.

17. Formulas for the Approximate Calculation of a Distributive Operator of f(x). In Section 11 the expression $\int_a^b \zeta(x)dx$ was taken as an approximation of $\int_a^b f(x)dx$, and a theory was developed for obtaining formulas of the type $\int_a^b \zeta(x)dx$ from known values of distributive operators of $f(x)$ at certain given points; however, the only property of the integral which is essential to this development is its distributive property. It is thus evident that if D is any distributive operator such that $D\phi(x)$ is either another function, as in the case where $D = \frac{d}{dx}$, or a constant, as in the case where $D = \int_a^b (\ )dx$; then if in Section 11 we replace the symbol $\int_a^b (\ )dx$ by D, the theory of this section applies in full to the derivation of formulas for the approximation of $Df(x)$ from known values of distributive operators of $f(x)$ at certain given points. Similarly, the theory of Sections 12, 13, 14, and 15 applies in every detail to simplify the derivation, to limit the error, to establish the uniqueness, and to investigate the invariant properties of such formulas.

MUTUAL INDUCTANCE AND REPULSION OF TWO ADJACENT DISK COILS

By H. B. Dwight and T. Y. Lu

In this paper are presented two formulas, the first, eq. (1), for the mutual inductance of two very thin, disk-shaped, duplicate, coaxial coils, and the second, eq. (2), for the electromagnetic force between the coils.

A convergent series for the mutual inductance of two coaxial circles was given by Maxwell.[1] This was extended to a considerable number of terms by E. B. Rosa and L. Cohen.[2] By the double integration of the formula for two circles, formula (1) for two disk coils is obtained. It is shown that for certain shapes and positions of the coils, particularly when they are close together, this formula has distinct advantages.

The mutual inductance of two thin, flat, coaxial coils, Fig. 1, is

$$M = \frac{\pi m N_1 N_2}{10^9}\left[\left(\log_n \frac{16m^2}{s^2+c^2}\right)\left(1 + \frac{c^2}{24m^2} + \frac{3}{4}\frac{s^2}{m^2} + \frac{11}{2880}\frac{c^4}{m^4} + \frac{7}{64}\frac{c^2s^2}{m^4} - \frac{15}{64}\frac{s^4}{m^4}\right)\right.$$
$$+ \left(\frac{s^2}{c^2}\log_n \frac{s^2+c^2}{s^2}\right)\left(1 + \frac{5}{8}\frac{s^2}{m^2} + \frac{5}{8}\frac{c^2s^2}{m^4} - \frac{161}{576}\frac{s^4}{m^4}\right)$$
$$- 4\left(\frac{s}{c}\tan^{-1}\frac{c}{s}\right)\left(1 + \frac{2}{3}\frac{s^2}{m^2} + \frac{2}{9}\frac{c^2s^2}{m^4} - \frac{2}{5}\frac{s^4}{m^4}\right)$$
$$\left. - 1 + \frac{43}{144}\frac{c^2}{m^2} + \frac{37}{24}\frac{s^2}{m^2} + \frac{c^4}{75m^4} - \frac{c^2s^2}{720m^4} - \frac{301}{360}\frac{s^4}{m^4}\right] \text{henrys} \quad (1)$$

[1] Electricity and Magnetism, Vol. II, Para. 705, and Sci. Paper 169 of the Bureau of Standards, eq. (10).

[2] Bulletin of the Bureau of Standards, Vol. 2, 1906, eq. (11), p. 366, and Sci. Paper No. 169, eq. (14).

where m is the mean diameter of the coils,
c is the radial depth of the coils,
s is the axial spacing,
N_1 and N_2 are the numbers of turns in the two coils and where the dimensions are in centimeters.

A very complete collection of mutual inductance formulas is given in Scientific Paper 169 of the Bureau of Standards, by E. B. Rosa and F. W. Grover. Four formulas applicable to disk-shaped coils are given and their characteristics are compared.

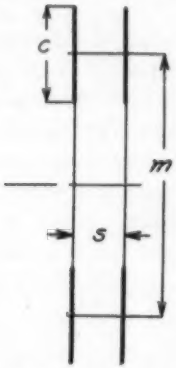

FIG. 1.—Two Adjacent, Thin, Disk Coils.

Three of these formulas, namely, (21) due to Rowland, (24) due to Rayleigh, and (28) due to Lyle, are for coils at a considerable distance apart, and the formulas become more accurate, the greater the distance between the coils.

A fourth formula, No. (29) by E. B. Rosa, contains terms in c^2s^2/m^4 and b^2s^2/m^4, using the notation of this paper. Consequently, as stated on page 40 of Sci. Paper 169, formula (29) becomes less exact as s is greater. However, it does not follow that (29) is suitable for small values of s in many cases. It contains a series of terms in c^2/s^2, c^4/s^4, c^6/s^6, $\cdots$, and accordingly it is unsuitable for very small values of s such as would be obtained in the case of two thin, flat coils close together, as in Fig. 1.

Formula (29) contains terms of the 2d, 4th and 6th degrees. In order that the formula should give a precise result, the terms of the 6th degree should be practically negligible and should be distinctly smaller than similar terms of the 4th degree. Some terms of the 6th degree, namely those in $1/a^6$, are not included in this formula, a being the mean radius, and so the terms in $1/a^4$ should be practically negligible.

While it is desirable that each user of a formula determine the degree of precision being obtained by watching the values of terms of different degrees, an approximate idea of the range in which ordinary accuracy can be obtained is given in Fig. 2.

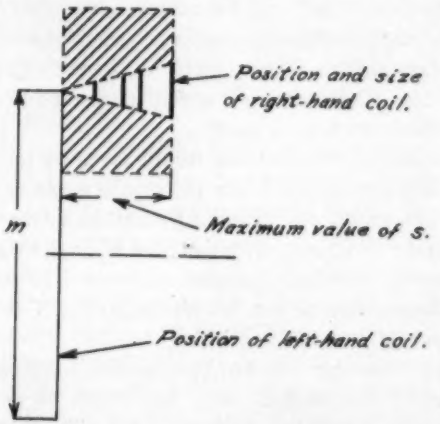

FIG. 2.—Ranges of Application of Formulas, for Two Equal Disk Coils.

For Eq. (29) of Sci. Paper 169, the ratio c/s should not be greater than about 1/2 and the ratio s/m should not be greater than about 1/3. The cross-section of the right hand coil should therefore lie within the unshaded cone shown in Fig. 2. A few possible sizes are indicated.

Formula (1) of this paper contains no terms with s in the denominator, and so the smaller s is, the more precise is the result. The ratio s/m should not be greater than about 1/3 and

c/m should not be more than about 1/2. The increase in the range which can be covered is shown by the shaded area in Fig. 2.

In using formula (1), terms of the 4th degree should be practically negligible and should be distinctly smaller than those of the 2d degree.

A direct check on some of the terms of formula (1) can be obtained by putting $s = 0$, in which case it becomes a self-inductance formula, namely, the formula for thin disk coils derived by T. R. Lyle.[3] It is allowable, in this way, to add two terms to formula (1), namely a term $103c^6/(105 \times 1024m^6)$ to the bracket following logn $16m^2/(s^2 + c^2)$ and a term $98579c^6/(88200 \times 512m^6)$ to be added after $(301/360)(s^4/m^4)$. These terms may improve the accuracy when the coils are very close together and s/m is very small. However, in general the additional terms are not very useful unless the corresponding terms in c^4s^2/m^6 etc. are available.

A comparison of the numerical results given by formulas (1) of this paper and (29) of Sci. Paper 169 can be made in cases where both series are rapidly convergent. For instance, when $s/m = 1/4$, $c/s = 1/2$ and $b = 0$, both formulas give $M = 1.737\pi mN_1N_2$.

When the coils are closer together, as $s/m = 1/10$ and $c/s = 1/2$, more significant figures can be obtained. In this case, both formulas give $M = 3.3893\pi mN_1N_2$.

Formula (1) has been checked by a laboratory measurement on two duplicate disk-shaped coils for which $m = 23.56$ cm., $c = 8.32$ cm., $s = 4.68$ cm., and $N = 516$. The tested value of mutual inductance was 0.0374 henry. The calculated value by (1) is 0.0366 henry, the difference being 2 per cent.

By differentiating (1) with respect to s, the following formula for the force exerted by the coils on each other, is obtained: Force, in dynes,

$$= \frac{\pi N_1N_2I_1I_2}{100}\left[\left(\log_n \frac{16m^2}{s^2 + c^2}\right)\left(\frac{3}{2}\frac{s}{m} + \frac{7}{32}\frac{c^2s}{m^3} - \frac{15}{16}\frac{s^3}{m^3}\right)\right.$$

[3] See *Phil. Trans.*, Vol. 213A, 1914, page 421, Sci. Pa. 320 of the Bureau of Standards, page 557 and Sci. Pa. 455, eq. (26).

$$+\left(\log_n \frac{s^2+c^2}{s^2}\right)\left(\frac{2ms}{c^2}+\frac{5}{2}\frac{s^3}{mc^2}+\frac{5}{2}\frac{s^3}{m^3}-\frac{161}{96}\frac{s^5}{m^3c^3}\right)$$

$$-4\left(\tan^{-1}\frac{c}{s}\right)\left(\frac{m}{c}+\frac{2s^2}{mc}+\frac{2}{3}\frac{cs^2}{m^2}-\frac{2s^4}{m^3c}\right)$$

$$+\frac{2ms}{(s^2+c^2)}\left(1+\frac{7}{12}\frac{s^2}{m^2}-\frac{c^2}{24m^2}+\frac{s^2}{c^2}+\frac{5}{8}\frac{s^4}{m^2c^2}\right.$$

$$\left.-\frac{11}{2880}\frac{c^4}{m^4}+\frac{193}{576}\frac{c^2s^2}{m^4}+\frac{19}{320}\frac{s^4}{m^4}-\frac{161}{576}\frac{s^6}{m^4c^2}\right)$$

$$\left.-\frac{2ms}{c^2}+\frac{37}{12}\frac{s}{m}-\frac{5}{4}\frac{s^3}{mc^2}-\frac{c^2s}{360m^3}-\frac{827}{180}\frac{s^3}{m^3}+\frac{161}{288}\frac{s^5}{m^3c^2}\right] \quad (2)$$

where I_1 and I_2 are in amperes. Note that in this formula only ratios of dimensions are involved. Large coils have larger forces than small ones of the same proportions, merely because the currents are larger.

Formula (2) was checked by a laboratory measurement on the two coils previously described, the force being 119.5 grams when $s = 4.68$ cm. and the current was 1.42 amperes. The calculated value of force is 107 grams, which is 10 percent less than the measured value.

Other tests are described in the thesis by T. Y. Lu, Massachusetts Institute of Technology, 1929.

It is possible that the formulas described in this paper may be of use in connection with the type of power transformers which are made up of flat disk coils, since the leakage inductance and the force in such transformers is only slightly affected by the presence of the iron core, and so it is possible to get useful accuracy from formulas based on non-magnetic cores.

DIFFERENTIAL PROJECTIVE GEOMETRY OF A SYSTEM OF PFAFFIAN DIFFERENTIAL EQUATIONS

By C. L. E. Moore

1. Introduction. Segre [1] investigated, in a projective space of n dimensions, the infinitesimal properties of curves, on a surface, which pass through a fixed point. In particular he showed that the locus of all the osculating planes at that point is a quadric cone of four dimensions having the tangent plane for vertex and lying in a space of five dimensions. He then investigated the properties of those surfaces for which this cone becomes a linear space of four dimensions. When this happens the coördinates satisfy a linear partial differential equation of the second order. The system of integral curves of a set of $n - 1$ Pfaffian equations, in a projective space of n dimensions, have many properties similar to those on a surface, and this note is concerned with an investigation similar to that of Segre. We find that the osculating planes of the integral curves which pass through a fixed point generate a quadric cone of four dimensions. When this cone becomes a linear space of four dimensions the system of integral curves resemble Segre's surfaces (those whose coördinates satisfy a partial differential of the second order).

The study of the geometry of the Pfaffian was first made by Abbé Issally [2] who used the name pseudo-surfaces. He showed the existence of geodesic lines, lines of curvature, asymptotic lines and conjugate systems. In 1901, Dall Aqua [3] attacked the problem by use of the absolute calculus and for a Riemannian space of three dimensions.

[1] Su una classa di superficie degl' iperspazii, atti di Torino, 1907.

[2] Pseudo-Surface, Bulletin de la Societe Math. de France 1888; Nouvelles Annales, 1900, and 1901.

[3] Sulla teoria delle congruenze di curve in una varieta qualunque, annali di matematica, 1901, Vol. VI, series III.

2. Tangent Plane and Osculating Spaces. In the case of non-homogeneous coördinates a system of Pfaffian equations are written

$$\sum_i A_{ij}dx^i = 0, \qquad j = 1, \cdots n-2, \tag{1}$$

but when we introduce homogeneous coördinates and make use of the usual relation connecting the coördinates

$$\sum_1^{n+1} x^i = 1, \tag{2}$$

we have, on differentiating, an additional differential equation. The system we shall discuss is that of $n-1$ equations in a projective space of n-dimensions. We then, can solve (1) and (2) explicitly for the differentials in terms of two of them. Thus

$$dx^\alpha = A_\alpha dx^1 + B_\alpha dx^2, \qquad \alpha = 1, 2, \cdots n+1, \tag{3}$$

where A_α, B_α are functions of all the variables x^i and satisfy the relations

$$A_1 = 1, \qquad A_2 = 0, \qquad B_1 = 0, \qquad B_2 = 1.$$

The system of integral curves of (3) we shall, for brevity, denote by φ, and use the expression curve of φ instead of an integral curve of (3). From (3) we see at once that all the curves of φ which pass through a given point x are tangent to the plane determined by the three points x^i, A_i, B_i. We then call the plane

$$|X\ x\ A\ B| = 0$$

the *tangent plane* to φ. A tangent line to φ at x is determined by the points

$$x,\ x + dx,$$

for which dx must satisfy (3).

If the three points

$$x,\ x + dx,\ x + 2dx + d^2x$$

lie on φ, then (3) and its differential must both be satisfied.

That is,

$$dx^\alpha = A_\alpha dx^1 + B_\alpha dx^2, \tag{3}$$

$$\begin{aligned} d^2x^\alpha &= \sum_\beta \frac{\partial A_\alpha}{\partial x^\beta} dx^\beta dx^1 + \sum_\beta \frac{\partial B_\alpha}{\partial x^\beta} dx^\beta dx^2 + A_\alpha d^2x^1 + B_\alpha d^2x^2 \\ &= \sum_\beta \frac{\partial A_\alpha}{\partial x^\beta} A_\beta (dx^1)^2 + \sum_\beta \left(\frac{\partial A_\alpha}{\partial x^\beta} B_\beta + \frac{\partial B_\alpha}{\partial x^\beta} A_\beta \right) dx^1 dx^2 \\ &\quad + \sum_\beta \frac{\partial B_\alpha}{\partial x^\beta} B_\beta (dx^2)^2 + A_\alpha d^2x^1 + B_\alpha d^2x^2. \end{aligned} \tag{4}$$

If we write

$$f = \sum \xi_i x^i = 0 \tag{5}$$

as the equation of a hyperplane passing through the point x, where ξ_i are independent of x, then the point x can be considered as given in hyperplanar coördinates. If the points $x + dx$, $x + 2dx + d^2x$, also lie in the hyperplane, then the first and second differentials of f also vanish. That is

$$f = \sum \xi_i x^i = 0, \tag{5}$$

$$\begin{aligned} \sum \xi_i (x^i + dx^i) &= \sum \xi_i dx^i \\ &= \sum \xi_i (A_i dx^1 + B_i dx^2) = f_1 dx^1 + f_2 dx^2 = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} \sum \xi_i (x^i + 2dx^i + d^2x^i) &= \sum \xi_i d^2x^i = \sum \xi_i \left[E_i (dx^1)^2 \right. \\ &\quad \left. + 2F_i dx^1 dx^2 + G_i (dx^2)^2 + A_i d^2x^1 + B_i d^2x^2 \right] = f_{11} (dx^1)^2 \\ &\quad + 2f_{12} dx^1 dx^2 + f_{22} (dx^2)^2 + f_1 d^2x^1 + f_2 d^2x^2 = 0, \end{aligned} \tag{7}$$

where

$$f_1 = \sum \xi_i A_i, \qquad f_2 = \sum \xi_i B_i,$$

$$f_{11} = \sum \xi_i E_i, \qquad f_{12} = \sum \xi_i F_i, \qquad f_{22} = \sum \xi_i G_i,$$

$$E_i = \sum_\beta \frac{\partial A_i}{\partial x^\beta} A_\beta, \qquad F_i = \frac{1}{2} \sum_\beta \left(\frac{\partial A_i}{\partial x^\beta} B_\beta + \frac{\partial B_i}{\partial x^\beta} A_\beta \right),$$

$$G_i = \sum \frac{\partial B_i}{\partial x^\beta} B_\beta.$$

By this representation the tangent plane to φ at a point x is given by the equations

$$f = 0, \qquad f_1 = 0, \qquad f_2 = 0. \tag{8}$$

These points cannot be collinear unless A_i and B_i are proportional. We will exclude this possibility from our consideration.

3. The osculating plane of a curve of φ is determined by the three points, x, $x + dx$, $x + 2dx + d^2x$. That is, by the three points

$$f = 0,$$

$$f_1 dx^1 + f_2 dx^2 = 0, \tag{9}$$

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 + f_1 d^2x^1 + f_2 d^2x^2 = 0.$$

We will first consider the locus of this plane when the tangent line is held fixed, that is, when $dx^1/dx^2 = k$. The third equation of (9) then becomes

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 + f_1(d^2x^1 - kd^2x^2) = 0,$$

and consequently the osculating plane is determined by the tangent line and some point on the line joining $f_1 = 0$ and

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 = 0. \tag{10}$$

Therefore the locus is an S_3 and since it is determined by the tangent line and the two points $f_1 = 0$, (10) we see that it is determined by the points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \tag{8}$$

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 = 0 \tag{10}$$

We then have the theorem: *The locus of the osculating planes of the curves of φ which issue from a fixed point in a fixed direction is an ordinary space of three dimensions which contains the tangent plane.*

If now we allow the tangent line to vary, the osculating planes will be the locus of the three space (8), (10) above, where dx^1/dx^2 is allowed to vary. If dx^1/dx^2 is allowed to vary, the locus of (10) is at once seen to be a conic. Then the desired locus is seen to be that of an S_3 which joins the tangent plane to the points of the conic (10) and is therefore a quadric cone V_4^2 of four dimensions. Hence: *The locus of the osculating planes of the curves*

of φ which pass through a fixed point is a quadric cone V_4^2 of four dimensions which has the tangent plane for vertex.

This cone lies in the five space determined by the six points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0,$$
$$f_{11} = 0, \quad f_{12} = 0, \quad f_{22} = 0.$$

Hence: *The points of φ near a fixed point to infinitesimals of the second order lie in a space of five dimensions.*

4. We will next investigate the locus of the osculating S_3 of the curves of φ which pass through a fixed point x. The hyperplane will then contain the four points

$$x, \quad x + dx, \quad x + 2dx + d^2x, \quad x + 3dx + 3d^2x + d^3x.$$

The expression for d^3x is obtained by differentiating (4) and substituting from (3). Using the above notation, we see that the hyperplane will contain the four points.

$$f = 0,$$

$$f_1 dx^1 + f_2 dx^2 = 0, \tag{12}$$

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 + f_1 d^2x^1 + f_2 d^2x^2 = 0,$$

$$\begin{aligned} f_{111}(dx^1)^3 + 3f_{112}(dx^1)^2dx^2 + 3f_{122}dx^1(dx^2)^2 + f_{222}(dx^2)^3 \\ + (2f_{11}dx^1 + A_{12}dx^2)d^2x^1 \\ + (A_{21}dx^1 + 2f_{22}dx^2)d^2x^2 + f_1 d^3x^1 + f_2 d^3x^2 = 0, \end{aligned} \tag{13}$$

where $f_1, f_2, f_{11}, f_{12}, f_{22}$ have the same meaning as before and

$$f_{111} = \sum_{i,\beta} \xi_i \frac{\partial E_i}{\partial x^\beta} A_\beta, \qquad f_{112} = \tfrac{1}{3} \sum_{i,\beta} \xi_i \left(\frac{\partial E_i}{\partial x^\beta} B_\beta + 2 \frac{\partial F_i}{\partial x^\beta} A_\beta \right),$$

$$f_{112} = \tfrac{1}{3} \sum_{i,\beta} \xi_i \left(\frac{2}{\partial x^\beta} \frac{\partial F_i}{\partial x^\beta} B_\beta + \frac{\partial G_i}{\partial x^\beta} A_\beta \right), \qquad f_{222} = \sum_{i,\beta} \xi_i \frac{\partial G_i}{\partial x^\beta} B_\beta,$$

$$A_{12} = \sum_{i,\beta} \xi_i \frac{\partial A_i}{\partial x^\beta} B_\beta, \qquad A_{21} = \sum_{i,\beta} \xi_i \frac{\partial B_i}{\partial x^\beta} A_\beta.$$

First hold the osculating plane fixed (i.e. hold dx^1, dx^2, d^2x^1, d^2x^2 constant) and consider the locus when d^3x^1, d^3x^2 are allowed to vary. As before we can write $dx^1/dx^2 = k$ and $f_2 = -kf_1$.

Then the last term of (13) becomes $f_1(d^2x^1 - kd^2x^2)$ and consequently when $(d^2x^1 - kd^2x^2)$ varies (13) will generate the line joining the points $f_1 = 0$ to the point obtained by equating the remainder of (13) to zero. Then the locus will join the osculating plane to the points of this line and therefore generate an S_4. This S_4 is then determined by the points

$$(8) \qquad f = 0, \quad f_1 = 0, \quad f_2 = 0,$$

$$(10) \qquad f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 = 0,$$

$$(14) \quad f_{111}(dx^1)^3 + 3f_{112}(dx^1)^2 + 3f_{122}dx^1(dx^2)^2 + f_{222}(dx^2)^3 + (2f_{11}dx^1 + A_{12}dx^2)d^2x^1 + (A_{21}dx^1 + 2f_{22}dx^2)d^2x^2 = 0.$$

Thus we see that the S_4 contains the tangent plane and the locus the osculating planes in the fixed direction dx^1/dx^2. If now d^2x^2 are allowed to vary (14) will describe the plane determined by the three points

$$(15) \quad f_{111}(dx^1)^3 + 3f_{112}(dx^1)^2dx^2 + 3f_{122}dx^1(dx^2)^2 + f_{222}(dx^2)^3 = 0,$$

$$(10_1) \qquad 2f_{11}dx^1 + A_{12}dx^2 = 0, \qquad A_{21}dx^2 + 2f_{22}dx^2 = 0.$$

Finally the complete locus will be described by the S_6 joining the seven points (8), (10), (10_1), (15). If dx^1/dx^2 is allowed to vary (10) will describe a conic (15) a cubic, (10_1), two ranges of points, all projectively related by means of the parameter dx^1/dx^2. The locus will then be a variety V_7^7 of degree seven. Hence the theorem: *The locus of the osculating 3-spaces to the curves of φ which pass through a fixed point x will generate a V_7^7 containing a simply infinite set of linear spaces of six dimensions.* This V_7^7 evidently lies in the 10-space determined by the eleven points

$$f = 0, f_1 = 0, f_2 = 0, f_{11} = 0, f_{22} = 0, A_{12} = 0, A_{21} = 0, f_{111} = 0, f_{112} = 0, f_{122} = 0, f_{222} = 0.$$

If $A_{12} = A_{21}$, the set of Pfaffians (3) are integrable and the above loci become those discussed by Segre [4]

In this case

$$f_{12} = A_{12} = A_{21}$$

[4] Loc. cit.

and (10_1) become

$$f_{11}dx^1 + f_{12}dx^2 = 0, \qquad f_{12}dx^1 + f_{22}dx^2 = 0.$$

The line joining these two points is seen to be tangent to the conic (10) and the locus is at once seen to be that of the S_5 joining the points

$$f = 0, \qquad f_1 = 0, \qquad f_2 = 0,$$
$$f_{11}dx^1 + f_{12}dx^2 = 0, \qquad f_{12}dx^1 + f_{22}dx^2 = 0,$$
$$f_{111}(dx^1)^3 + 3f_{112}(dx^1)^2dx^2 + 3f_{122}dx^1(dx^2)^2 + f_{222}(dx^2)^3 = 0.$$

This locus is a V_6^5 which agrees with Segre's result.

5. Singularities. We will now turn our attention to the singularities of the intersection of φ with the hyperplanes passing through the point x. That is, the curves of φ which lie in a hyperplane through x. Let the hyperplane be

$$f = \sum \xi_i x^i = 0.$$

If we indicate a point of φ near to x and lying in f, by

$$x + dx + \frac{1}{2}d^2x + \frac{1}{6}d^3x + \cdots$$

and substitute from equation (4) we have

$$\begin{aligned} &f_1dx^1 + f_2dx^2 \\ &+ \frac{1}{2}\left[f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2\right] \\ &+ \frac{1}{6}\left[f_{111}(dx^1)^3 + 3f_{112}(dx^1)^2dx^2\right. \\ &\qquad \left. + 3f_{122}dx^1(dx^2)^2 + f_{222}(dx^2)^3\right] + \cdots \\ &+ f_1d^2x^1 + f_2d^2x^2 + (2f_{11}dx^1 + A_{12}dx^2)d^2x^1 \\ &\qquad + (A_{21}dx^1 + 2f_{22}dx^2)d^2x^2 + \cdots = 0. \end{aligned}$$

If f_1 and f_2 do not vanish then the direction of the curve of intersection is given by

$$f_1dx^1 + f_2dx^2 = 0,$$

and the curve has a simple point at x.

If the hyperplane passes through the tangent plane, then $f_1 = 0, f_2 = 0$ and the directions of the curve of intersection are given by

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 = 0, \tag{10}$$

and therefore the curve of intersection has a double point whose tangents are determined by the values of dx^1/dx^2 satisfying the above equation.

If the hyperplane passes through the space

$$f = 0, f_1 = 0, f_2 = 0, f_{11} = 0, f_{12} = 0, f_{22} = 0, A_{12} = 0, A_{21} = 0,$$

then the curve of intersection will have a triple point at x and the directions of the tangents are given by the values of dx^1/dx^2 satisfying

$$f_{111}(dx^1)^3 + 3f_{112}(dx^1)^2dx^2 + 3f_{122}dx^1(dx^2)^2 + f_{222}(dx^2)^3 = 0.$$

In a similar manner multiple points of higher order can be obtained.

If a curve of φ, passing through x and lying in f has a double point with distinct tangents then the osculating planes to the two branches of the curve will lie in the V_4^2. These are the planes in which f cuts this cone. If

$$f_{11}f_{22} - f_{12}^2 = 0,$$

the double point will have coincident tangents. The hyperplane will then be tangent to the cone V_4^2.

6. Spaces of Three Dimensions. If we limit ourselves to a space of three dimensions the system of equations (3) reduces to two. The tangent line to a curve of φ passing through x, is the line joining the points

$$f = 0, \qquad f_1dx^1 + f_2dx^2 = 0.$$

For this case we can determine the direction of a tangent line which will have three nearby points of φ on it. Let three nearby points be

$$f = 0, \qquad f_1dx^1 + f_2dx^2 = 0,$$

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 + f_1d^2x^1 + f_2d^2x^2 = 0.$$

It is evident that the tangent line which contains these three points will be the lines joining x to the points of intersection of the tangent plane with the conic

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 = 0.$$

The directions dx^1/dx^2 for which this happens is given by

$$|xABf_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2| = 0$$

or

$$D(dx^1)^2 + 2D'dx^1dx^2 + D''(dx^2)^2 = 0 \tag{16}$$

where

$$D = |x\ A\ B\ E|,$$
$$D' = |x\ A\ B\ F|,$$
$$D'' = |x\ A\ B\ G|,$$

These directions are the *asymptotic* directions.[5]

The polar form of (16)

$$Ddx^1\delta x^1 + D'(dx^1\delta x^2 + dx^2\delta x^1) + D''dx^2\delta x^2 = 0 \tag{17}$$

determines the conjugate directions having the property of conjugate directions of surfaces in ordinary 3-space. That is, the tangent planes to φ at points x, $x + dx$ will intersect in a line whose direction $\delta x^1/\delta x^2$ satisfies (17). This can be proved in the usual way.

7. Degeneration of the Cone V_4^2. So far we have considered only the case where the locus of the osculating planes of curves of φ passing through a fixed point is a non-degenerate cone V_4^2 and now we wish to consider what happens in case this cone becomes a linear space of four dimensions. This will happen when the six points (1,1) saitsfy a linear relation i.e., lie in an S_4. The cone then lies in this S_4 and being of four dimensions will coincide with the S_4 itself. Let the linear relation be

$$aE_\alpha + bF_\alpha + cG_\alpha + eA_\alpha + gB_\alpha + hx^\alpha = 0. \tag{18}$$

Multiplying these equations by ξ_α and summing on α we have

$$af_{11} + bf_{12} + cf_{22} + ef_1 + gf_2 + hf = 0. \tag{19}$$

[5] Dall' Aqua, note 3.

In a space S_4, of four dimensions, we see that φ will always satisfy (18) and in a space S_3, of three dimensions, two such equations are always satisfied. For a space of higher dimensions (18) will be satisfied when and only when the determinants of the matrix

$$\| E \; F \; G \; A \; B \; x \|$$

vanish.

For example, if (3) is of the form

$$dx^\alpha = \varphi_\alpha(x^1, x^2 \cdots x^{n+1})dx^1 + \psi_\alpha(x^1)dx^2$$

where

$$\varphi_1 = 1, \qquad \psi_1 = 0, \qquad \varphi_2 = 0, \qquad \psi_2 = 1.$$

Since

$$G_\alpha = \sum \frac{\partial \psi_\alpha}{\partial x^\beta} \psi_\beta$$

we see that equation (18) takes the simple form

$$G_\alpha = 0.$$

In this case, through each point passes a straight line which belongs to the system φ.

If equations (3) have the form

$$dx^\alpha = \varphi_\alpha(x^2)dx^1 + \psi_\alpha(x^1)dx^2,$$

where

$$\varphi_1 = 1, \qquad \psi_1 = 0, \qquad \varphi_2 = 0, \qquad \psi_2 = 1,$$

it is seen at once that two equations

$$E_\alpha = 0, \qquad G_\alpha = 0$$

are satisfied.

We will now limit ourselves to the case where there is a single equation (18) satisfied by the quantities E_α, F_α, G_α, A_α, B_α, x^α. As we saw, this relation can be written in the form (19) and if we take a hyperplane which passes through the tangent plane, the tangents at the double point, in x, are given by (10). Now (19) is the condition that the directions given by (10) separate harmonically the directions determined by

$$c(dx^1)^2 - bdx^1dx^2 + a(dx^2)^2 = 0. \tag{20}$$

These latter directions depend only on the coefficients of (18) and not on the particular hyperplane passing through the tangent plane. Hence we have the theorem: *The pair of tangents, at a point x, to the curves of φ having a double point in x are not all the pairs to tangents through x, but only the pairs of an involution whose double lines are given by (20).*

The two directions of (20) are those for which the double tangents coincide. The directions of (20) we will call the *characteristic directions* and the envelope of these directions, the *characteristics* of φ.

The curves of φ, which pass through a point x, form a surface whose coördinates $x^1(u, v)$, $x^2(u, v)$, at the point x, satisfy the equation

$$(18') \qquad ax_{uu}{}^i + bx_{uv}{}^i + cx_{vv}{}^i + ex_u{}^i + gx_v{}^i + hx^i = 0,$$

because at this point all the osculating planes to the curves of φ lie in an S_4 and consequently, the curves of the surface passing through x have the same property. Equation (18′) is the condition that this should happen. Consequently, at this point the surface will have the same characteristic directions as φ.

The tangent plane at a point x is determined by the three points,

$$x,\ A,\ B,$$

and the tangent plane at the nearby point $x + dx$, where dx is a direction in the tangent plane, is determined by the three points

$$x + Adx^1 + Bdx^2,\quad A + Edx^1 + Fdx^2,\quad B + Fdx^1 + Gdx^2.$$

If equation (18) is satisfied these two planes will lie in an S_4. Let us take one of the coördinate 4-spaces to coincide with the S_4. The condition that these two nearby tangent planes should have a line in common i.e., lie in an S_3 is that dx^1/dx^2 be so determined that the points

$$Edx^1 + Fdx^2, \qquad Fdx^1 + Gdx^2$$

lie in the tangent plane at x. This condition is

$$[x,\ A,\ B,\ Edx^1 + Fdx^2,\quad Fdx^1 + Gdx^2] = 0,$$

or

$$[x\,A\,B\,E\,F](dx^1)^2 + [x\,A\,B\,E\,G]dx^1dx^2 + [x\,A\,B\,F\,G](dx^2)^2 = 0.$$

If now from (18) we solve for the ratios of the coefficients we get

$$a : b : c = [x\,A\,B\,F\,G] : -[x\,A\,B\,E\,G] : [x\,A\,B\,E\,F]$$

and therefore, the above condition becomes

$$c(dx^1)^2 - bdx^1dx^2 + a(dx^2)^2 = 0. \tag{20}$$

This shows that the tangent planes at two nearby points on a characteristic intersect in a line or that tangent planes at all points of a characteristic form a developable surface.

If we choose coördinates so that at x the directions of the characteristics coincide with the x^1 and x^2 directions, we have $a = 0$, $c = 0$. From this we see that if b does not vanish, the point F must lie in the tangent plane, i.e., F is linearly expressible in terms of x, A, B. The tangent plane at the nearby point in the direction of the characteristic will be determined by the three points.

$$x + Adx^1, \qquad A + Edx^1, \qquad B + Fdx^1.$$

It is then obvious that the first and last points lie in the tangent plane at x. So $dx^1 \to 0$, we see then that the line of intersection of the two planes is the line joining x and B. This is the direction of the second characteristic through x. We then have the theorem:

If the system φ satisfies (18) there is then a double system of curves, two through each point, such that the tangent planes at points of a curve of one set form a developable surface whose generators are tangent to the other set.

8. Parabolic Case. If equation (18) satisfies the relation

$$b^2 - 4ac = 0, \tag{21}$$

we will call the equation of parabolic type. In this case, the two systems of characteristics coincide and this system of characteristics have the characteristic property of asymptotic lines of a surface in ordinary space of three dimensions, viz. tangent planes

at two nearby points of a characteristic intersect in a tangent line to the characteristic.

We will next find the condition that a tangent plane to φ will contain three nearby points. Let the three points be given by (5), (6), (7). The tangent plane to φ at the point x is defined by $f = 0$, $f_1 = 0$, $f_2 = 0$. If the curve of which (5), (6), (7) are three nearby points is to have the tangent plane for osculating plane it is necessary that the tangent plane contain the point

$$f_{11}(dx^1)^2 + 2f_{12}dx^1dx^2 + f_{22}(dx^2)^2 = 0. \tag{10}$$

For the points (5), (6) and $f_1d^2x^1 + f_2d^2x^2 = 0$ already lie in the tangent plane. Then the condition that the tangent plane contain the points (5), (6), (7) is

$$f_{11}k^2 + 2kf_{12} + f_{22} + ef_1 + gf_2 + hf = 0, \tag{22}$$

where

$$k = dx^1/dx^2.$$

This condition is both necessary and sufficient. Hence: *If a parabolic equation of type (22) is satisfied, then through each point passes a curve of φ whose osculating plane coincides with the tangent plane, and conversely.*

Now two conditions are necessary for φ to have a tangent line at each point having three point contact with φ. (*a*) The point (10) must lie in the tangent plane; (*b*) for some choice of d^2x^1/d^2x^2 the point (7) must lie on the tangent line. The point (7) lies on the line joining (10) to

$$f_1d^2x^1 + f_2d^2x^2 = 0,$$

and consequently if (22) is satisfied, then when d^2x^1/d^2x^2 varies, the point (7) will describe a line in the tangent plane. This line will cut the tangent line drawn in the direction dx^1/dx^2. Thus, if condition (*a*) is satisfied, condition (*b*) is satisfied also. Hence: *The necessary and sufficient condition that φ have, through each point, a tangent line having three point contact with φ is that a parabolic equation of type (22) be satisfied.*

From the form of (22) we see that the directions of the tangents with three point contact coincide with the direction of the characteristic through the point.

DYNAMICAL SYSTEMS WITH INTEGRALS QUADRATIC IN THE VELOCITIES [1]

By Philip Franklin

Motions of a particle in a plane under conservative forces possessing an integral, other than that of energy, quadratic in the velocities may be found by a method due to Bertrand,[2] which leads to a second order partial differential equation. This equation was integrated by Darboux,[3] who showed that in the general case, the only systems of this type are problems of Liouville's type, provided that the parametric curves are taken as suitable ellipses and confocal hyperbolas. In the present note, the special cases left out are investigated, and it is found that they merely lead to the limiting cases of the confocal ellipses and hyperbolas, i.e. confocal parabolas, the parametric curves for polar coördinates, or those for ordinary Cartesian coördinates.

The connection of our result with the notion of conditional integral [4] introduced by Birkhoff, and a theorem of his on conditional integrals is indicated.

For the sake of completeness, we recall the argument of Darboux.[3] As our system is conservative, the forces are derivable from a potential energy function V, and the equations of motion may be written

$$(1) \quad mx'' = -\partial V/\partial x = -V_x, \quad my'' = -\partial V/\partial y = -V_y.$$

We seek systems which possess an integral of the form

$$(2) \quad Px'^2 + Qx'y' + Ry'^2 + Sy' + Tx' + K/m = \text{constant},$$

[1] Presented to the American Mathematical Society at Bethlehem, Dec. 27, 1929.

[2] Journal de Math., (i), XVII, (1852), p. 121.

[3] Archives Néerlandaises, (ii), VI, (1901), p. 371. Darboux's note is essentially reproduced in Whittaker's Analytical Dynamics, (1904), p. 321.

[4] Birkhoff, Dynamical Systems, 1927. (A. M. S. Colloquum, vol. IX), pp. 45, 48.

where P, Q, R, S, T, K are functions of x and y. On differentiating (2), and using (1), we find the relation:

$$(3)\quad x'^3P_x + x'^2y'(P_y + Q_x) + y'^2x'(Q_y + R_x) + y'^3R_y + x'^2T_x + x'y'(T_y + S_x) + y'^2S_y + x'/m(K_x - 2PV_x - QV_y) + y'/m(K_y - QV_x - 2RV_y) - 1/m(TV_x + SV_y) = 0.$$

As x, y, x', y' may be taken arbitrarily, this equation is an identity in all the variables, and we may equate the coefficients of the terms in the primed variables to zero. The first four give:

$$(4)\quad P_x = 0,\quad P_y + Q_x = 0,\quad Q_y + R_x = 0,\quad R_y = 0.$$

Hence:

$$-Q_{xy} = P_{yy} = R_{xx} = 2a,$$

since P involves y only while Q involves x only. Thus:

$$(5)\qquad P = ay^2 + by + c,\qquad R = ax^2 + dx + e,$$
$$Q = -2axy - bx - dy + f,$$

a, b, c, d, e, f being constants.

The second degree terms in (3) lead to:

$$(6)\qquad T_x = 0,\qquad S_y = 0,\qquad T_y + S_x = 0.$$

Hence:

$$T_y = -S_x = g,$$

since T involves y only while S involves x only. Thus:

$$(7)\qquad S = -gx + h,\qquad T = gy + k,$$

g, h, k being constants.

The terms of (3) independent of x' and y' give in view of (7),

$$(8)\qquad (gy + k)V_x + (-gx + h)V_y = 0.$$

If $g = 0$, we may take the point h/g, $-k/g$ as a new origin, and reduce this equation to the form:

$$yV_x - xV_y = 0,$$

whose solution is

$$(9)\qquad V = f(x^2 + y^2).$$

If $g = 0$, while h and k are not both zero, we may take new coördinate axes parallel to the lines $hy + kx = 0$, $hx - ky = 0$, and reduce equation (8) to the form:

$$V_y = 0,$$

whose solution is

$$V = f(x). \tag{10}$$

If $g = h = k = 0$, our integral contains no terms linear in the velocities. In other cases, the terms linear in the velocities are themselves an integral, and (9) and (10) are in accord with the general theorem [6] that any dynamical system which possesses an integral linear and homogeneous in the p_i, possesses an ignorable coördinate, if the proper extended point transformation is made. In our case we have the theorem:

THEOREM I. *Any motion of a particle in a plane under conservative forces, possessing an integral linear in the velocities, possesses an ignorable coördinate θ, when a suitable system of polar coördinates is introduced, or an ignorable coördinate y, when a suitable system of Cartesian coördinates is introduced.*

In the cases just considered, the terms of (2) quadratic in the velocities may be any linear combination of the square of the linear integral, and the kinetic energy.

We now turn to the more general case, where there is no integral linear in the velocities. Here $S = T = 0$, so that we have merely equation (5) and the conditions derived from the first degree terms of (3):

$$K_x - 2PV_x - QV_y = 0. \quad K_y - QV_x - 2RV_y = 0. \tag{11}$$

The condition of integrability for this pair of equations in K is that the values of K_{xy} and K_{yx} as derived from the two equations be equal, or:

$$2PV_{xy} + 2V_xP_y + QV_{yy} + Q_yV_y = 2RV_{xy} + 2R_xV_y + Q_xV_x + QV_{xx}. \tag{12}$$

[6] Cf. Whittaker, p. 318.

In view of (5), this becomes:

$$(13)\quad (V_{yy} - V_{xx})(-2axy - bx - dy + f) + 2V_{xy}(ay^2 - ax^2 + by - dx + c - e) + V_x(6ay + 3b) + V_y(-6ax - 3d) = 0.$$

We shall simplify this by a change of coördinates, and notice that the quadratic integral, by (5) is:

$$(14)\quad a(yx' - xy')^2 + (bx' - dy')(yx' - xy') + cx'^2 + fx'y' + ey'^2 + K/m = \text{constant}.$$

When $a \neq 0$, we have the case treated by Darboux. We notice that when the origin is shifted to the point $-d/2a$, $-b/2a$, the second group of terms in (14) drops out. We may then rotate the axes so as to be the principal axes of the conic

$$(15)\quad cx^2 + fxy + ey^2 = 1,$$

using the c, f, and e from the reduced form, and this will drop out the term in $x'y'$, while it will not affect the first term since $(yx' - xy')$ is invariant under a rotation. Thus the integral is reduced to

$$(16)\quad a(yx' - xy')^2 + cx'^2 + ey'^2 + K/m = \text{constant}.$$

If we introduce a new constant p, by

$$(17)\quad c - e = ap^2,$$

the partial differential equation (13) takes the form:

$$(18)\quad xy(V_{xx} - V_{yy}) + (y^2 - x^2 + p^2)V_{xy} + 3yV_x - 3xV_y = 0.$$

The differential equation for the characteristics is

$$(19)\quad xy(dy^2 - dx^2) + (x^2 - y^2 - p^2)dxdy = 0.$$

This may be integrated by noting that when x^2 and y^2 are taken to be the variables, it is a Clairaut equation. When p is not equal to zero, the integral curves are the confocal ellipses and

hyperbolas given by:

$$x^2/A^2 + y^2/(A^2 - p^2) = 1. \tag{20}$$

On taking the parameters of these confocals, A and B as new variables, we have

$$x = AB/p, \qquad y = 1/p\sqrt{(A^2 - p^2)(p^2 - B^2)}, \tag{21}$$

and equation (18) takes the form:

$$(B^2 - A^2)V_{AB} + 2BV_A - 2AV_B = 0. \tag{22}$$

This is readily integrated, successively, and gives as the integral:

$$(A^2 - B^2)V = f(A) - g(B). \tag{23}$$

From (21), we find that the kinetic energy,

$$T = m/2(x'^2 + y'^2) = \frac{m}{2}(A^2 - B^2)\left[\frac{A'^2}{(A^2 - p^2)} + \frac{B'^2}{(p^2 - B^2)}\right], \tag{24}$$

and a comparison of this with the potential energy function, given by equation (23),

$$V = \frac{f(A) - g(B)}{A^2 - B^2}, \tag{25}$$

shows that the motion is of Liouville's type.

When p is equal to zero, the integral curves of (19) are the lines through the origin, and the concentric circles orthogonal to them, whose equations may be written:

$$x^2 + y^2 = r^2, \qquad y = x \tan \theta. \tag{26}$$

We take the parameters of these curves, polar coördinates, as the new variables, so that

$$x = r \cos \theta, \qquad y = r \sin \theta, \tag{27}$$

and transform equation (18), with p equal to zero, into:

$$rV_{\theta r} + 2V_\theta = 0. \tag{28}$$

On integrating this twice, we find for the integral,

$$r^2V = f(r) - g(\theta). \tag{29}$$

Since, in this case, we may write the kinetic energy in the form:

$$T = \frac{mr^2}{2}\left[\frac{r'^2}{r^2} + \theta'^2\right], \tag{30}$$

and the potential energy function, as given by (29), is:

$$V = \frac{f(r) - g(\theta)}{r^2}, \tag{31}$$

the motion is of Liouville's type in this case, also.

We next consider the reduction of the integral (14) to simpler form when a is equal to zero. We assume, for the present, that b and d are not both equal to zero, and notice that in this case a rotation of the axes which makes the y axis parallel to the line $bx - dy = 0$ will, in effect, make $d = 0$ and $b \neq 0$. We may then shift the origin to the point f/b, $(e - c)/b$ and the integral (14) will assume the form:

$$bx'(yx' - xy') + e(x'^2 + y'^2) + K/m = \text{constant}. \tag{32}$$

The partial differential equation (13) is in this case:

$$x(V_{xx} - V_{yy}) + 2yV_{xy} + 3V_x = 0. \tag{33}$$

The differential equation for the characteristics is

$$xdy^2 - 2ydxdy - xdx^2 = 0. \tag{34}$$

This may be integrated, either from its homogeneity, or by noting that it is a Clairaut equation when x^2 is taken as a new variable in place of x. The integral curves are the confocal parabolas given by:

$$x^2 + 2Ay - A^2 = 0. \tag{35}$$

On introducing the parameters of these confocals, A and B, as new variables, we have

$$x = \sqrt{-AB}, \qquad y = (A + B)/2, \tag{36}$$

and equation (33) is transformed into:

$$V_{AB}(A - B) - V_A + V_B = 0. \tag{37}$$

This is readily integrated, successively, and we find for the integral:

$$(A - B)V = f(A) - g(B). \tag{38}$$

We may obtain the kinetic energy from (36), and so write

$$T = \frac{m}{2}(x'^2 + y'^2) = \frac{m}{2}(A - B)\left[\frac{A'^2}{A} - \frac{B'^2}{B}\right]. \tag{39}$$

This, compared with the potential energy function of (38),

$$V = \frac{f(A) - g(B)}{A - B}, \tag{40}$$

shows the motion to be again of Liouville type.

We have left the investigation of the integral (14) when a, b and d are all zero. We may make f equal to zero, by rotating the axes into parallelism with the principal axes of the conic (15), and the integral (14) is then reduced to the simpler form:

$$cx'^2 + ey'^2 + K/m = \text{constant}. \tag{41}$$

If c and e were equal, this would reduce to the energy integral, so that we may assume

$$c - e \neq 0, \tag{42}$$

and write as the partial differential equation (13) in this case, $V_{xy} = 0$, so that

$$V = f(x) + g(y). \tag{43}$$

Thus the motion is again of Liouville type.

Our discussion is now complete, since all possible cases have been treated. We obtained the expressions (25), (31), (40) and (43) by excluding the cases where an integral linear in the velocities existed. We may, however, remove this restriction, since the earlier forms (9) and (10) are special cases of the forms (31) and (43). We summarize our results in:

THEOREM II. *Any motion of a particle in a plane under conservative forces, possessing an integral quadratic in the velocities distinct from the energy integral is of Liouville's type when a suitable family of parametric curves is introduced.*

Depending on which of four types of integral is present, the proper parametric curves will be a family of confocal ellipses and hyperbolas, a family of confocal parabolas, the lines and circles of a polar coördinate system, or the lines of a rectangular Cartesian coördinate system.

The forms for the kinetic and potential energy are given in equations (24), (25), (30), (31), (39), (40), and (43).

A well known example of the elliptic case is the motion of a particle attracted to two fixed centers according to the inverse square law, and also acted on by an attracting center midway between the other two centers and in line with them attracting according to the direct distance law, as well as forces parallel to the join of the two first points and their perpendicular bisector, each of these last two sets of parallel forces being inversely proportional to the cube of the distance from the other line. In this case we have:

$$V = \frac{s}{\sqrt{(x-p)^2+y^2}} + \frac{t}{\sqrt{(x+p)^2+y^2}} + u(x^2+y^2) + \frac{v}{x^2} + \frac{w}{y^2}. \tag{44}$$

If the coördinates of (21) are introduced, the expression just written is the special case of (25) with:

$$\begin{aligned} f(A) &= (s+t)A - up^2A^2 + uA^4 - vp^2/A^2 + wp^2/(A^2-p^2), \\ g(B) &= (t-s)B - up^2B^2 + uB^4 - vp^2/B^2 + wp^2/(B^2-p^2). \end{aligned} \tag{45}$$

It is easy to give special examples of the other types, for which the forces have a simple physical interpretation. Thus if a particle is acted on by a center attracting according to the inverse square law, and in addition is acted on by two systems of forces parallel to two perpendicular axes through this center, in

the one case inversely as the cube of the distance to the other line, and in the other case by a constant force, we have the parabolic case. Here

$$V = \frac{t}{\sqrt{x^2 + y^2}} + \frac{v}{x^2} + wy. \tag{46}$$

If the coördinates of (36) are introduced, the expression just written is the special case of (40) with:

$$\begin{aligned} f(A) &= wA^2/2 + 2t + v/A, \\ g(B) &= wB^2/2 + v/B. \end{aligned} \tag{47}$$

A special example of the polar coördinate case would be a particle attracted to a fixed center by any function of the distance law, and also acted on by any number of sets of forces, each set being perpendicular to a fixed line through the origin, and inversely proportional to the cube of the distance to this line. The Cartesian case is simply that in which the particle is acted on by forces parallel to two fixed, perpendicular lines, the force parallel to either line being a function of the distance to the other line.

As the problems we are discussing all have only two degrees of freedom, and possess two integrals, namely that of energy, and the one under consideration, they are soluble by quadratures This is also evident from the fact that they are of the Liouville type. We may deduce an interesting converse fact. For, if a system is of Liouville type, it is known to have an integral quadratic in the velocities Consequently, if it is conservative, it is of the type we are considering, and we have:

THEOREM III *The only motions of a particle in a plane under conservative forces, which are of Liouville's type, are of this type when referred to a suitable system of elliptic, parabolic, polar or Cartesian coördinates*

Birkhoff [4] has defined as a "conditional" integral of a dynamical system, an integral which only holds for one particular value of the energy constant. For conservative systems with two degrees of freedom, he shows that when a conditional integral

linear in the velocities is at hand the system can be "transformed" into a system with an ignorable coördinate, for which the potential energy function is given by (10) above, while when a conditional integral quadratic in the velocities is at hand the system can be "transformed" into a system for which the potential energy function is given by (43) above. These results at first sight seem to be in contradiction to ours, since they do not take account of our remaining types, and in the sense in which we have used the notion of transformation in this paper, our types can not be transformed one into another.

The explanation of the paradox is the type of transformation used by Birkhoff. Having fixed the energy constant, he confines his attention to the set of trajectories belonging to that constant, so that his "transformations" of the system are transformations of a subset of trajectories of the system.

It is interesting that the extension of the notion of integral, to that of conditional integral holding for a particular value of the energy constant, does not lead to anything essentially new in the case of integrals linear and quadratic in the velocities. In such cases, the set of trajectories for that value of the constant are transformable, under a change of coördinates and time whose effect on the remaining trajectories of the system is not taken account of, into a set of trajectories for a particular value of the energy constant, of the two simplest types of system we have found above.

As an example, consider the dynamical system for which

$$T = F(A, B)(A'^2 + B'^2), \qquad V = -1/F(A, B). \tag{48}$$

The problem of finding the trajectories for a given energy constant, h, where:

$$T + V = h, \tag{49}$$

is the energy integral is equivalent [6] to that of the geodesics for

[6] Cf. Birkhoff, l. c., p. 39; Whittaker, l. c., p. 249.

the surface with linear element:

$$(50) \qquad ds^2 = (h + 1/F)F(dA^2 + dB^2), \\ = (1 + hF)(dA^2 + dB^2).$$

Unless $F(A, B)$ is of special form,[7] for values of h different from zero this arc element will correspond to a surface whose geodesic problem has no integrals linear or quadratic in the velocities. When, however, h equals zero, we have the linear integrals:

$$(51) \qquad FA' = c_1, \qquad FB' = c_2,$$

connected by the energy relation

$$(52) \qquad c_1^2 + c_2^2 = 1.$$

For *this value of the energy constant*, the trajectories map on the straight lines in a plane, traversed with constant velocity, under the type of transformation used by Birkhoff.

The geodesic representation of reversible dynamical systems,[6] at once reduces problems on conditional integrals to similar problems on geodesics, and the normal forms for reversible dynamical systems with two degrees of freedom are consequences of the geometric theorems,[7] that a surface whose geodesic problem possesses a linear integral is applicable to a surface of revolution, while when a quadratic integral is at hand, distinct from the "energy" quadratic integral, the surface is either one whose element of arc can be put into the form of Liouville:

$$(53) \qquad ds^2 = [f(A) - g(B)](dA^2 + dB^2),$$

or the form:

$$(54) \qquad ds^2 = [Af(B) + g(B)]dAdB.$$

The second type only gives rise to a real surface when it is a special case of the first, and appears when the two quadratic integrals have a common linear factor. In Birkhoff's discussion,

[7] G. Darboux, Lecons sur la Theorie generale des Surfaces, Paris, 1894, vol. 3, pp. 29, 33, or F. Massieu, Paris Thesis, 1861.

he is dealing with real dynamical systems, with kinetic energy reducible to the form

$$x'^2 + y'^2$$

by real transformations, so that to have this case the additional quadratic integral would have to contain the factor $x \pm iy$. The case escapes when he makes the assumption, justifiable for real dynamical systems, that terms can only combine by containing the factor $(x'^2 + y'^2)$; in systems (54) they combine in virtue merely of containing linear factors of this expresson.